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Geometric transitions, brane dynamics and gauge theories

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Abstract: We consider the interplay between brane constructions and type IIA, IIB or M-theory geometries on Calabi-Yau (CY) and G_2 holonomy manifolds. This is related to $\mathcal{N}=1$ (and $\mathcal{N}=2$) gauge theories in four dimensions. We first discuss simple geometric transitions corresponding to brane set ups involving orthogonal (or parallel) Neveu-Schwarz branes that approach each other. This is related to confinement and Seiberg duality in SQCD. In particular, we argue that in type IIA, a $\mathbb{CP}^1$ of singularities and one unit of Ramond-Ramond (RR) flux is dual to a D6 brane wrapped on a Lens space, describing the UV and IR of $\mathcal{N}=1$ Super-Yang-Mills (SYM), respectively. Also, in the large N_c duality that relates D6 branes on S^3 to an S^2 with RR flux, we implement the presence of N_f flavors of quarks. We then compactify M-theory on $(T^*(S^3)\times S^1)/\mathbb{Z}_2$ and observe that one phase describes SO(4) SYM in the UV and two others describe confinement. Moreover, we consider compact 7-spaces $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$. We describe transitions where disconnected S^3's approach and connect each other before they vanish. These effects correspond to non-abelian Higgs mechanism and confinement. The similar transitions involving $S^2\times S^1$'s are also considered. Finally, we present transitions at finite distance in moduli space, where the first Betti number b_1 of 3-cycles of singularities changes.

Keywords: D-branes, M-Theory, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

The interplay between gauge theories, brane dynamics and geometry is a useful tool to learn about non-trivial phenomena in one of these frameworks since sometimes it is manifest in one of the other descriptions (see [1] for a review on brane dynamics and gauge theory, and [2] for reviews on geometrical engineering). For instance, various properties of brane dynamics can be deduced by known properties of the gauge theory living on the branes at low energies and vice versa. On the other hand, non-trivial dualities in gauge theory — like the Seiberg's duality [3] in Supersymmetric Yang-Mills (SYM) — are manifested as deformations in the space of brane configurations [4], providing a "unification" of the dualities for different gauge groups with various matter content and in various dimensions in a single framework.

There are however subtleties in the study of gauge dynamics using branes. The "rules" governing the behavior of branes, especially in transitions involving coincident branes, are not always clear. In particular, the transitions due to coincident Neveu-Schwarz (NS) fivebranes involve the non-trivial theory on their worldvolume. Hence, generically, one is not guaranteed to have a smooth transition when parallel NS branes approach each other. On the other hand, the crossing of orthogonal NS branes is leading to the $\mathcal{N}=1$ electric-magnetic duality in four dimensions, and thus is expected to be smooth. An extensive discussion of transitions due to intersecting NS branes appears in [1, section IX B2].

Confinement can also be described by two orthogonal flat NS branes intersecting in $1+3$ dimensions, in the presence of RR flux. Quantum gauge theory effects will turn out to bend the branes classically.

In many cases, known properties in certain gauge theories allow to set rules of brane dynamics, which can then be used in more complicated systems. Sometimes, such rules can be confirmed by standard perturbative worldsheet considerations [5]. Alternatively, known properties in geometry can also be used [6] to shed light on the kinematics and dynamics of brane constructions as well as gauge dynamics. In this work we focus on the latter.

The frameworks employed here are M-theory on G_2 holonomy manifolds and orientifolds of type IIA on Calabi-Yau (CY) threefolds. The low energy physics is four dimensional and is $\mathcal{N}=1$ supersymmetric. The geometry on CY threefolds is accompanied by D-branes wrapped on non-trivial cycles and/or RR fluxes. On the contrary, the description in the framework of G_2 7-manifolds is purely geometric.

Since there is no non-abelian structure in eleven dimensions to start with, compactifications on smooth 7-manifolds give rise to abelian vector multiplets. However, it is known that the existence of singularities in the internal space leads to non-abelian gauge theories. In the M-theory framework, it has been shown in [7] that, in certain cases, as 2-cycles collapse in the internal space, an A_n singularity is developed giving rise to an enhancement of the gauge group. This is the mechanism employed in [8] for the study of $\mathcal{N}=1$ six dimensional gauge theories in the AdS/CFT correspondence. From the type IIA point of view, the enhancement of the gauge group is due to coincident D6 branes, the description of which is purely geometric from the M-theory point of view [9].

In this work, we argue that smooth transitions corresponding to intersecting NS branes translate in geometry into topology changes at finite distance in moduli space, and vice versa. To investigate this interplay, we begin in section 2.1 with a review of relatively simple brane configurations describing $\mathcal{N}=1$ SQCD and $\mathcal{N}=2$ SYM, and geometries obtained from them by performing various chains of T-dualities. In particular, the two possible small resolutions of the conifold by a 2-sphere S^2 or $\tilde{S}^2$ give rise to a flop transition $S^2 \to 0 \to \tilde{S}^2$. In type IIB, when D5-branes are wrapped on them, this transition is associated to electric-magnetic duality. In section 2.2, we review various geometric constructions involving special lagrangian (SLAG) 3-cycles on which D6 branes are wrapped in type IIA. In particular, it has been argued that a stack of N_c D6 branes wrapped on S^3 is dual to an S^2 with N_c units of Ramond-Ramond (RR) flux through it [10]. We shall see that in this geometric set up, which describes confinement of pure SU(N_c) SYM, one can include N_f massless flavors of quarks. Section 2.3 is devoted to the lift of such type IIA descriptions to M-theory. First, the case of orbifolds of the non-compact G_2 holonomy manifold Spin(S^3) is recalled [11, 12]. We then treat the case of $(T^*(S^3)\times S^1)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts antiholomorphically on $T^*(S^3)$ and as an inversion on S^1. We shall see that there are three distinct phases in this model. The first one is associated to SO(4) SYM in the UV, while the other two describe confinement. The situation is similar to the three phases occuring in the models based on Spin(S^3) [13]. Also, we propose a new duality conjecture. It will be argued that in type IIA on a CY threefold, a $\mathbb{CP}^1$ of A_{N_c-1} singularity and one unit of RR charge through it is dual to a Lens space $S^3/\mathbb{Z}_{N_c}$ with one D6 brane wrapped on it. These two phases describe the UV and IR physics of the pure $\mathcal{N} = 1$ SU(N_c) gauge theory, respectively.[1] Finally, an M-theory background Spin(S^3)/($\mathbb{Z}_{N_f-N_c} \times \mathbb{Z}_{N_f}$) is considered as a candidate for describing the UV and IR physics of the magnetic SU($N_f - N_c$) SYM with N_f flavors of quarks. Work related to issues discussed in section 2 appear also in [14, 15, 16].

We then study transitions in compact manifolds or singular spaces of G_2 holonomy. Even if we choose a particular construction of the latter, the local geometry of the transitions we shall describe explicitly can then occur in other G_2 manifolds. Precisely, we shall consider orbifolds of the form (CY$\times S^1$)/$\mathbb{Z}_2$. The $\mathbb{Z}_2$ acts again as an inversion on S^1 and antiholomorphically on the CY, so that $J \to -J$ and $\Omega \to \bar{\Omega}$, where J and Ω are the Kähler form and holomorphic 3-form [17]. In general, the fixed point set Σ of the antiholomorphic involution on the CY is then composed of disconnected special lagrangian 3-cycles. Each componant in Σ is promoted to an associative 3-cycle of A_1 singularities in the 7-space. The first Betti number b_1 of such an associative 3-cycle then counts the number of chiral multiplets in the adjoint representation of an SU(2) gauge group. The model presented in section 3.1, describes a theory without massless adjoint fields. In the underlying CY, disconnected 3-spheres approach each other till we reach a transition where they intersect at a point. Then, they are replaced by the connected sum $S^3 \# S^3$, which is topologically equivalent to a single S^3. Transitions of this type might be related to the work of Joyce [18]. [2] Physically, in M-theory and a type IIA orientifold limit, this amounts to the Higgsing SO(4) $\times$ SO(4) $\to$ SO(4) by a massless (4, 4) chiral multiplet. As discussed in section 3.2, this Higgs branch *does not* exist classically in field theory and is due to a dynamically generated superpotential. This non-perturbative effect is described in the brane construction by the classical bending of intersecting orthogonal NS branes. In other phases, confinement of SO(4)^2 or SO(4) also take place, as in [12, 20]. From the brane point of view, these transitions are smooth, while from the geometric point of view, they are argued to occur at finite distance in moduli space, like standard conifold transitions between CY's.

In section 4.1, the effects of a CY conifold transition [21] on M-theory compactified on (CY$\times S^1$)/$\mathbb{Z}_2$ are considered. This amounts to S^3's of A_1 singularities in M-theory that undergo flop transitions to $\mathbb{RP}^3$'s with no singularity. In a type IIA orientifold limit, this is described by a transition $S^3 \to \mathbb{RP}^2$, with no RR flux. It is interpreted as confinement together with a change of branch in the scalar potential of neutral chiral multiplets, as in [20, 22]. These mixed effects are in contrast with the pure confining phenomenon occuring in the non-compact cases based on Spin(S^3) and $T^*(S^3)$. Then, it is suggested that an

[1] And similarily for D and E Lie groups.

[2] In [19], disjoint homology 3-spheres in different classes are connected to each other after a transition occurs. In our case, the 3-spheres are in the same class.

orientifold of type IIA on a compact CY with a 3-sphere of singularities might be dual to a type IIA orientifold on the mirror CY. The former describes SYM in the UV, while the latter could describe the IR. In section 4.2, we consider a situation where a non-abelian Higgs mechanism takes place together with a change of branch in the scalar potential. On one side of the transition, this is described by D6 branes wrapped on two disconnected S^3's, each intersecting a third homology 3-sphere between them at a point. At the transition, this third 3-cycle undergoes a conifold transition to an S^2, in the spirit of [23].

Finally, in section 5, we consider two very different situations where 3-cycles with non-trivial b_1 are involved. The models are still of the form $(\mathrm{CY}\times S^1)/\mathbb{Z}_2$. In section 5.1, we focus on the case where each S^3 occuring in section 3.1 is replaced by $S^2 \times S^1$. This amounts to having an adjoint field of the gauge group and hence a Coulomb branch. As an example, one of the transitions involves a disconnected union $(S^2 \cup S^2) \times S^1$ that is becoming $(S^2 \# S^2) \times S^1 \cong S^2 \times S^1$. In general, such transitions concern the non-holomorphic S^2's that are part of 3-cycles and are expected to occur at infinite distance in moduli space. In the brane picture, they correspond to parallel NS branes (in the presence of RR background) that approach each other. On the contrary, in section 5.2, we consider transitions at finite distance in moduli space where the first Betti number b_1 of 3-cycles changes. Explicitely, we describe a situation where four disconnected 3-spheres centered at the corners of a square approach each other till they intersect at four points, giving rise to a non-contractible loop passing through these points. In total, the transition takes the form $\bigcup_{i=1}^4 S^3 \to \#_{i=1}^4 S^3 \cong S^2 \times S^1$. Then, the radius of the S^1 can also decrease and we pass into a third phase: $S^2 \times S^1 \to S^3$, with again a jump in b_1. In these cases, both the brane picture and the field theory interpretation deserve to be studied further.

2. Simple brane constructions, geometry and SYM

2.1 Type IIA branes versus type IIB geometry

We first discuss systems which describe four dimensional SQCD at low energy. Consider a brane configuration in the type IIA string theory constructed out of NS fivebranes and Dirichlet fourbranes (D4) (later, we shall also study examples that include orientifold four-planes (O4)) whose worldvolume is stretched in the directions:

$$\begin{aligned} \mathrm{NS} &: (x^0, x^1, x^2, x^3, x^4, x^5) \\ \mathrm{NS'} &: (x^0, x^1, x^2, x^3, x^8, x^9) \\ \mathrm{D4/O4} &: (x^0, x^1, x^2, x^3, x^6)\,. \end{aligned} \tag{2.1}$$

We separate the NS and NS' branes a distance L_c in the direction x^6 and stretch N_c D4 branes between them (see figure 1a); we shall call these branes the "color branes." The low energy theory on the D4 branes is an $\mathcal{N} = 1$, $\mathrm{SU}(N_c)$ SYM in the $1 + 3$ dimensions common to all the branes in eq. (2.1). The gauge multiplet corresponds to the low lying excitations of open strings stretched between the D4 branes.

We will sometimes also add N_f semi-infinite D4 branes stretched along the direction x^6 on the other side of the NS' brane (see figure 1b); we shall call these branes the "flavor

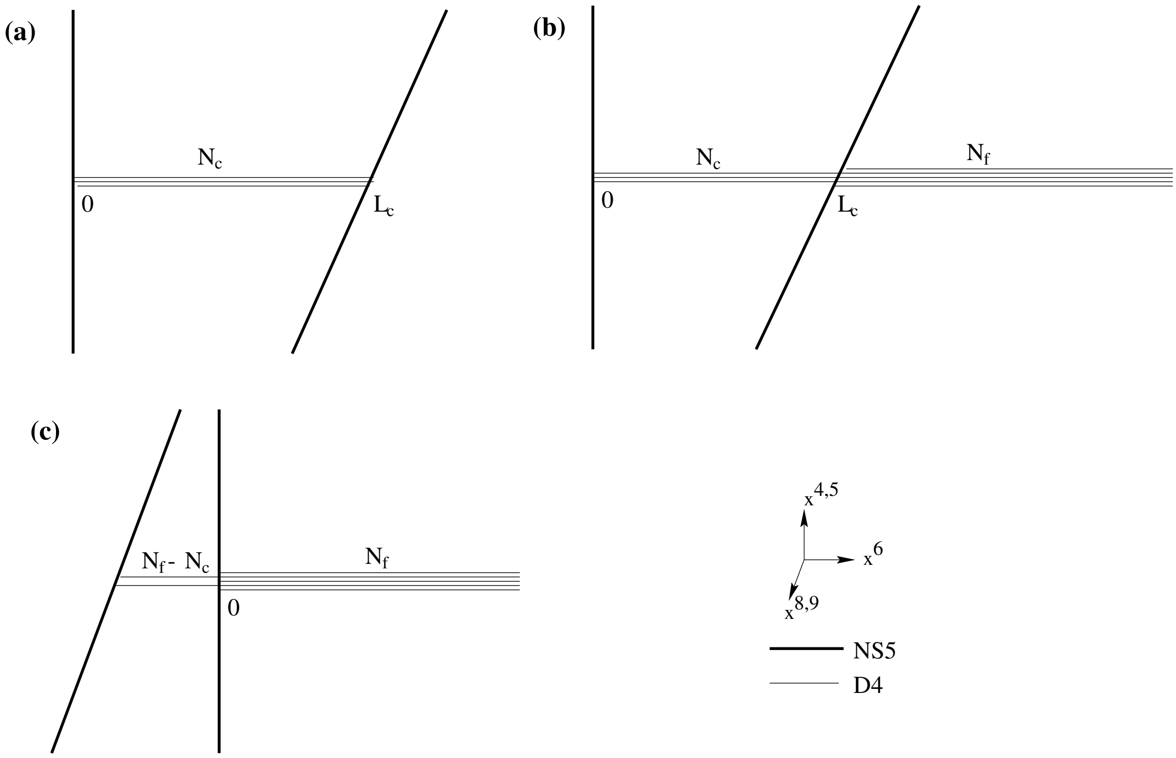

Figure 1: (a) Realization of pure SU(N_c) SYM with branes. (b) Flavors of quarks are introduced by semi-infinite D4 branes. (c) The magnetic dual description of (b) is obtained by interchanging the positions of the Neveu-Schwarz branes.

branes." The SU(N_f) corresponding to open strings stretched between the flavor branes is a global symmetry from the point of view of the SU(N_c) gauge theory on the color branes. Open strings stretched between the color and flavor branes correspond to (N_c, N_f) hypermultiplets in SU$(N_c) \times$ SU(N_f), and thus give rise to N_f quark and anti-quark chiral multiplets in the SU(N_c) gauge theory. While this can be guessed from the fact that in the vicinity of the NS brane the system has an $\mathcal{N} = 2$ supersymmetry, it cannot be deduced in worldsheet perturbation theory due to the fact that open strings confined to the vicinity of the NS brane are strongly coupled.[3]

We may also consider a system where x^6 is compactified on a circle of radius R_6. In this case we cannot have semi-infinite flavor branes, but instead we can stretch N_f D4 branes of length $2\pi R_6 - L_c$ along the left and right side of NS and NS' branes, respectively (see figure 2a). The low energy theory is now an $\mathcal{N} = 1$, SU$(N_c) \times$ SU(N_f) gauge theory with an (N_c, N_f) hypermultiplet.

A third system whose low lying theory is $\mathcal{N} = 1$, SU$(N_c) \times$ SU(N_f) with a bi-fundamental hypermultiplet is described in figure 3a. It consists of two NS branes and one NS' brane between them, N_c D4 branes of length L_c stretched between the first NS and the NS', and N_f D4 branes of length L_f stretched between the NS' and the second NS brane. In the limit where the second NS brane is sent to infinity we obtain the system in figure 1b.

[3]For a separated stack of NS branes this was verified by standard worldsheet techniques in [5].

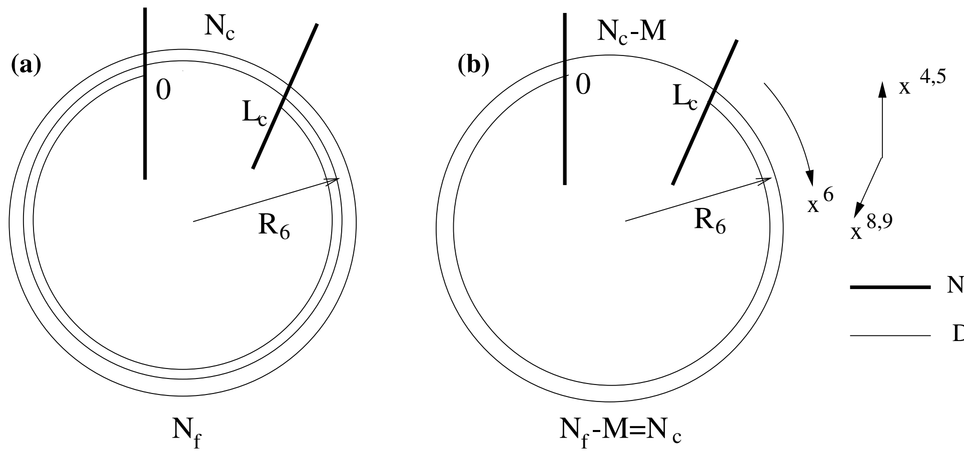

Figure 2: (a) Brane realization of $N = 1$, $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ with an (N_c, N_f) hypermultiplet. (b) The first step of a duality cascade.

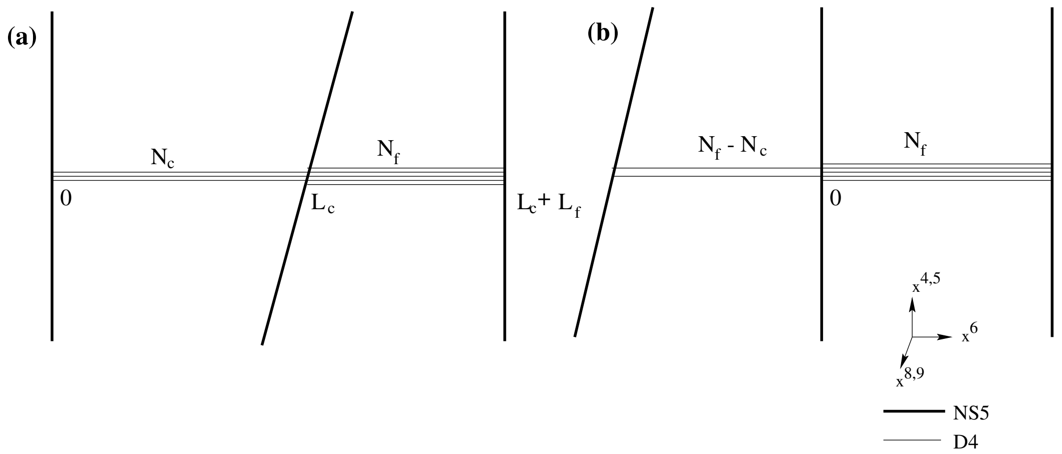

Figure 3: (a) Brane realization of the "electric" $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ SYM. (b) Realization of the "Magnetic" dual $\mathrm{SU}(N_f - N_c) \times \mathrm{SU}(N_f)$ theory.

In all these systems we can add an O4 plane parallel to the D4 branes (2.1). The sign of the RR charge of the orientifold is flipped each time it is separated in two by a single Neveu-Schwarz brane (see [1] for a review). When the RR charge of the orientifold is negative and there are N_c D4 branes on top of it[4] the low energy theory has an $\mathrm{SO}(N_c)$ gauge symmetry. On the other hand, when the RR charge of the orientifold is positive the gauge group is $\mathrm{Sp}(N_c/2)$.

For the system of figure 2a, in the limit $R_6 \to 0$, it is convenient to do a T-duality T_6 in the direction x^6, bringing the system to a type IIB description. In the limit where also $L_c/R_6 \to 0$, the NS and NS' system in figure 2a can be regarded as a Neveu-Schwarz fivebrane wrapped on the singular Riemann surface defined by the algebraic equation in $\mathbb{C}^2$:

$$H(v, w) = vw = 0\,, \quad \text{where} \quad v = x^4 + ix^5\,, \quad w = x^8 + ix^9\,. \tag{2.2}$$

[4]Our convention is that N_c is the *total* number of D4 branes: "Half" D4 branes together with their mirror partners under the orientifold reflection.

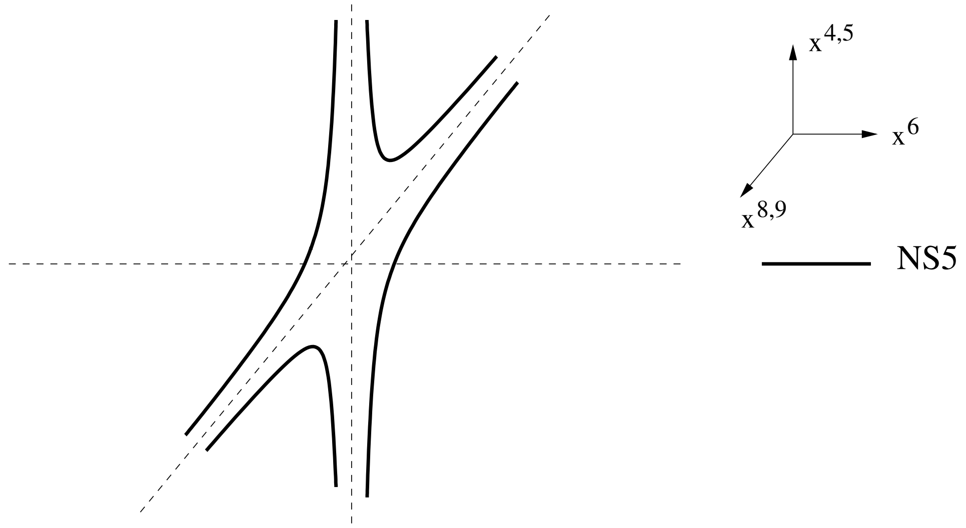

Figure 4: After two orthogonal Neveu-Schwarz branes intersect, they can bend. The gauge group has disappeared; it confines.

The T_6 duality turns the type IIA string theory in the presence of such a fivebrane into a type IIB theory on $\mathbb{R}^{1,3} \times \mathcal{C}$, where $\mathcal{C}$ is defined by the algebraic equation in $\mathbb{C}^4$:

$$F(v, w, z, z') = H(v, w) - zz' = vw - zz' = 0\,. \tag{2.3}$$

This singular six dimensional space is the conifold — a cone with an $S^2 \times S^3$ base. In the resolved conifold, there is a single non-trivial cycle — the blown up S^2. On the contrary, the deformed conifold has only a non-trivial S^3 (this is discussed further in section 2.2).

The relation between the two T-dual pictures is the following. The distance L_c/R_6 in figure 2a turns into a blow up parameter of S^2.[5] On the other hand, the deformed conifold

$$F(v, w, z, z') = H(v, w) - zz' = vw - zz' = \mu \tag{2.4}$$

is T-dual to a Neveu-Schwarz fivebrane wrapped on the Riemann surface

$$H(v, w) = vw = \mu\,. \tag{2.5}$$

The parameter μ in eq. (2.4) is related to the radius of S^3 in the deformed conifold. In the fivebrane picture, when μ is turned on, the coincident NS and NS' branes bend such that they do not pass through the origin $v = w = 0$ (see figure 4). Thus, there is no gauge group.

Next we discuss the T-dual description of the D4 branes in $\mathcal{C}$. For simplicity, consider first the case $N_f = 0$. The color D4 branes in figure 2a turn into N_c D5 branes wrapped on the S^2 cycle of the resolved conifold. The limit $L_c/R_6 \to 0$ is dual to the singular

[5] More precisely, there are two "blow up" modes of S^2: A θ parameter due to a two index B-field on S^2, and vol(S^2). The θ parameter is related to the separation $L_6 = L_c$ of the fivebranes in the x^6 direction and hence to the YM coupling (see below), while vol(S^2) is related to a separation L_7 in x^7 and hence to a FI D-term in the $\mathcal{N} = 1$ U(N_c) SYM (see [1]).

conifold where the S^2 cycle is degenerated and the D5 branes look like "fractional" D3 branes at the tip of the conifold. Since classically $L_c \sim 1/g_{YM}^2$, as we decrease L_c, we increase the YM coupling g_{YM}. Quantum mechanically, g_{YM} is dimensionally transmuted into a dynamically generated scale Λ; it runs towards strong coupling in the IR. Hence, L_c or its dual S^2 are dynamically degenerated. It is claimed [24, 10, 25] that when S^2 shrinks, the S^3 of the conifold dynamically blows up to a size related to the QCD scale Λ and that there are N_c units of RR flux through S^3. Hence, the geometrical conifold transition describes confinement of SU(N_c) SQCD. In the brane configuration of figure 2a (and similarly for the brane system of figure 1a), this transition corresponds to the limit when the NS and NS' branes intersect and bend away from the $v = w = 0$ point, as shown in figure 4, in the presence of RR flux.

A small number of flavors $N_f < N_c$ in figure 1b and 2a does not change the physics above: In type IIA, the intersection of the NS and NS' branes still amounts to confinement and should again be dual to the conifold transition $S^2 \to 0 \to S^3$ in type IIB. However, when $N_f \geq N_c$ another physical transition is possible, namely an $\mathcal{N} = 1$ electric-magnetic duality. Starting with the electric theory in figure 1b, this corresponds to changing the position of the NS' brane from positive to negative coordinate x^6. This gives rise to a magnetic dual $\mathcal{N} = 1$, SU($N_f - N_c$) SYM with N_f flavors (see figure 1c). Actually, when $3N_c > N_f \geq N_c$ this is expected to happen dynamically: The system is driven towards strong coupling where the NS and NS' branes intersect. Since physically on both sides of this transition we have a gauge theory with matter and it is a question of convention to decide which one is the electric or magnetic description,[6] we expect in the T-dual description of type IIB a transition involving on each side 2-spheres. In other words, the electric-magnetic duality should be realized as a geometric flop transition $S^2 \to 0 \to \tilde{S}^2$ in type IIB, where S^2 and $\tilde{S}^2$ are the two possible small resolutions of a conifold[7] (see the recent work [56]).

Similarly, in the system of figure 2a, when $N_c \geq N_f - N_c = M \geq 0$ the dynamics is expected to lead to the duality cascade [24]: $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c + M) \to \mathrm{SU}(N_c - M) \times \mathrm{SU}(N_c) \to \mathrm{SU}(N_c - 2M) \times \mathrm{SU}(N_c - M) \to \cdots \to \mathrm{SU}(N_c - kM) \times \mathrm{SU}(N_c + M - kM)$, until $N_c - kM < M$ (the first step of the cascade is shown in figure 2b). After reaching the last step, the theory confines. In the fivebranes description, this cascade is dynamically due to the bending of the NS and NS' branes (see [24] for details). On the type IIB side, this should correspond to the cascade of geometrical flops and conifold transition: $S^2 \to 0 \to \tilde{S}^2 \to 0 \to S^2 \to 0 \to \cdots \to S^2 \to 0 \to S^3$. On duality cascades and Seiberg duality, see also [15, 16].

In order to shed some light on the type IIB geometric description of electric-magnetic duality, it is useful to recall the IIA branes/IIB geometry duality relation for an $\mathcal{N} = 2$ configuration. Concretely, we consider a system of two parallel NS branes separated by a distance L_c in the direction x^6. Between them, N_c D4 branes are stretched, whose coordinates in the v-plane are v_i ($i = 1, \ldots, N_c$), describing the eigenvalues of the vacuum

[6] The detailed structure of the theories is missing for semi-infinite flavor D4 branes.

[7] Locally, near the singularity, the two resolutions are isomorphic, but global effects and/or the presence of D-branes can distinguish the two.

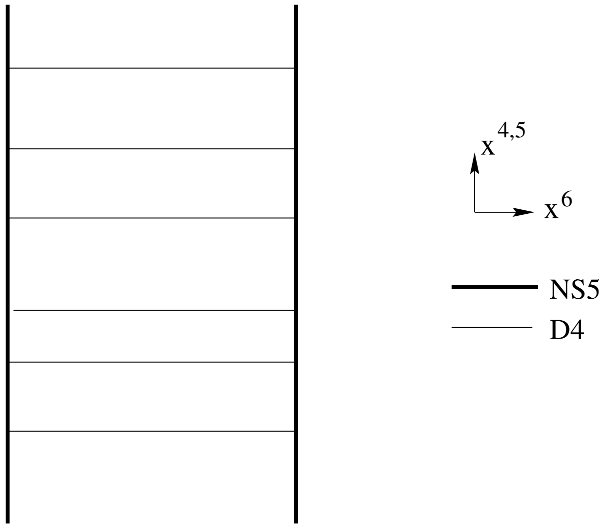

Figure 5: Brane realization of pure $\mathcal{N} = 2$ SU(N_c) SYM in the Coulomb branch.

expectation value (VEV) of an adjoint field of the SU(N_c) gauge theory in its Coulomb branch (see figure 5). After compactification of the x^6 direction, the function appearing in eq. (2.2) is now $H(v, w) = w^2$ and the variable v is not constrained. The algebraic equation (2.3) in $\mathbb{C}^4$ then becomes

$$F(v, w, z, z') = H(v, w) - zz' = w^2 - zz' = 0\,, \tag{2.6}$$

which describes the ALE space $\mathbb{C}^2/\mathbb{Z}_2$ times $\mathbb{C}$, the local geometry of a singular $K3 \times T^2$ compactification of type IIB. Now there is only one way to desingularize this space, namely by blowing up an S^2 at the origin of the ALE space:

$$F(v, w, z, z') = H(v, w) - zz' = w^2 - zz' = \mu\,. \tag{2.7}$$

This is T-dual in type IIA to a Neveu-Schwarz fivebrane on:

$$H(v, w) = w^2 = \mu\,, \tag{2.8}$$

namely, two NS branes separated by a distance $L_c/R_6 \sim \sqrt{\mu}$. As in the $\mathcal{N} = 1$ case, L_c/R_6 in type IIA is mapped to a blow up parameter of the S^2 in type IIB. Here, however, there is no physical (or geometrical) transition occurring when the NS branes are coincident (or the S^2 vanishes). Notice that the fact that the six dimensional geometry takes the form of a product $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ implies that we have actually a blown up S^2 at each point v in $\mathbb{C}$. Therefore, this S^2 has a one complex dimensional moduli space and the N_c wrapped D5 branes can independently sit at any points v_i ($i = 1, \ldots, N_c$) in $\mathbb{C}$, describing the Coulomb branch of an $\mathcal{N} = 2$ vector multiplet in the adjoint of SU(N_c).

Let us treat now in the same spirit the brane system of figure 3a. In order to obtain a geometrical description of this configuration, we compactify the direction x^6 on a circle of radius R_6 and perform a T-duality on it. On the type IIB side, the CY geometry

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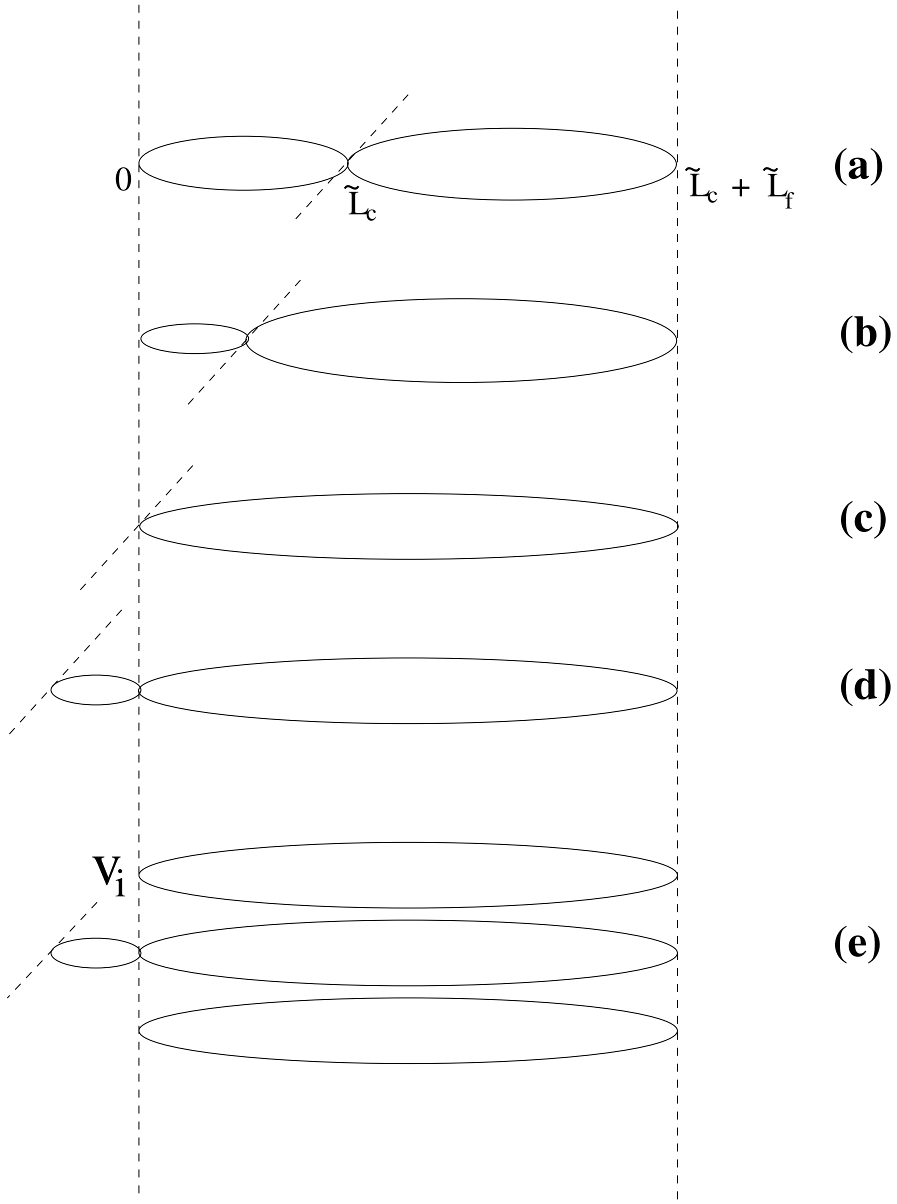

Figure 6: (a) Local type IIB geometry realizing $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ SYM, T-dual to figure $3a$ indicated in dashes. Only the 2-cycles where D5 branes are wrapped are represented. (b,c,d) Type IIB transition dual to the crossing of NS and NS' branes that describes Seiberg duality. (e) One of the $\mathbb{CP}^1$'s has now a moduli space that parametrizes the "meson" VEV.

resulting from the presence of the three type IIA NS branes is thus expected to contain two isolated 2-spheres with blow up parameters L_c/R_6 and L_f/R_6 and intersecting at a point, as depicted in figure $6a$.

Clearly, this geometry is compatible with the geometrical realization of the system of figure $1a$ or $1b$ in the limit $L_f/L_c \gg 1$. In addition, the type IIA D4 branes give rise to N_c (N_f) D5 branes wrapping the first (second) of these S^2's, generating an $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ gauge theory. The fact that the 2-spheres have to be isolated is required by the fact that there is no adjoint field in any of the gauge group factors. Also, the 2-spheres have to intersect for the (N_c, N_f) hypermultiplet to be massless. As a remark, the $\mathcal{N} = 2$ version of the previous brane configuration has already been considered in the literature [26]. It is obtained by replacing the middle NS' brane by a third parallel NS brane.[8] The dual type IIB geometric realization has been conjectured to consist of a pair of 2-spheres intersecting at a point as before, each of them living in a one complex dimensional family. As required, this geometry describes the Coulomb branch of an $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)$ $\mathcal{N} = 2$ gauge theory.

When $N_f \geq N_c$, let us describe what should be the geometrical picture related to the magnetic dual brane configuration of figure 3*b* obtained by moving the NS' brane from positive to negative values of its coordinate x^6. From the brane point of view, we now have an $\mathrm{SU}(N_f - N_c) \times \mathrm{SU}(N_f)$ gauge theory with a hypermultiplet $(q, \tilde{q})$ in the $(N_f - N_c, N_f)$ representation and a chiral field M in the adjoint of $\mathrm{SU}(N_f)$. The $\mathcal{N} = 2$ supersymmetry associated with the two parallel NS branes implies the existence of a classical superpotential $\mathcal{W} = qM\tilde{q}$. After a T_6 duality along the compact direction x^6, the type IIB configuration is expected to give an isolated 2-sphere, which we will denote by $\tilde{S}^2$, with $N_f - N_c$ wrapped D5 branes on it and intersecting at one point a second S^2 with N_f D5 branes on it (see figure 6*d*). However, this second S^2 should live in a one complex dimensional family of 2-spheres in the CY, so that the N_f D5 branes can slide on any other representative of the family. This would describe the Coulomb branch of the $\mathrm{SU}(N_f)$ gauge factor obtained by giving a vacuum expectation value to the "meson" field M, as depicted in figure 6*e*. In total, figure 6 illustrates a sequence of 2-spheres describing a geometrical transition from an electric to a magnetic description.

To summarize, the singular type IIB geometry T-dual to an NS brane intersecting an NS' brane at a point a finite distance away from another NS brane should admit the three different desingularizations described above. Two of them are the small resolutions where a 2-sphere S^2 or $\tilde{S}^2$ is blown up, while the third one consists in the appearance of a 3-sphere. In the next section, we shall describe the mirror picture of these 3 phases.

2.2 Mirror duality from type IIB to type IIA geometry

Mirror symmetry relates generically a CY threefold in type IIB to a different one in type IIA. In particular, it maps the resolved conifold in type IIB to the deformed conifold in type IIA, and vice versa. Hence, the conifold transition of the previous subsection,

$$\mathrm{IIB} \quad : \qquad S^2 \to 0 \to S^3 \,, \tag{2.9}$$

is mirror dual to:

$$\mathrm{IIA} \quad : \qquad S^3 \to 0 \to S^2 \,. \tag{2.10}$$

Recall that both transitions in eqs. (2.9), (2.10) are T-dual to the transition when NS and NS' branes approach, intersect and then bend.

To be concrete, let us describe explicitly the type IIA geometry on the deformed conifold $T^*(S^3)$, defined in eq. (2.4); here we write it in the form:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu \,, \tag{2.11}$$

[8]This amounts to take $H(v, w) = w^3$ in eq. (2.2).

where z_i ($i = 1, 2, 3, 4$) are complex coordinates and μ can be chosen to be real and positive without loss of generality. The base S^3 is identified with the fixed point set of the antiholomorphic involution $z_i \to \bar{z}_i$. Denoting by x_i and y_i the real and imaginary parts of z_i, it is automatically a special lagrangian submanifold, whose equation is:

$$y_i = 0\,, \qquad x_1^2 + x_2^2 + x_3^2 + x_4^2 = \mu\,, \tag{2.12}$$

showing that its volume is related to μ.

Let us concentrate first on the realization in this context of the pure SU(N_c) SYM theory. We saw in the previous section how the D4 branes of figures 1a turn into D5 branes wrapped on S^2 in type IIB. The mirror picture in type IIA thus involves N_c D6 branes wrapping the S^3 of eq. (2.12). The massless spectrum contains a U(N_c) gauge group.

At $\mu = 0$, the deformed conifold becomes a cone whose apex at $z_i = 0$ can be blown up to an S^2. The resolved conifold obtained this way can be written as

$$\begin{cases} z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \\ (z_3 - iz_4)\xi_1 + (z_1 - iz_2)\xi_2 = 0\,, \end{cases} \tag{2.13}$$

where $\xi_{1,2}$ are projective coordinates of $S^2 \cong \mathbb{CP}^1$. In this form, it is clear that any non-singular point on the conifold is lifted to a single point on the resolved conifold, while at the singular point $z_i = 0$ on the conifold, $\xi_{1,2}$ are not fixed and parametrize a full $\mathbb{CP}^1$. Alternatively, the resolved conifold can be rewritten as

$$\begin{cases} (z_1 + iz_2)\xi_1 - (z_3 + iz_4)\xi_2 = 0 \\ (z_3 - iz_4)\xi_1 + (z_1 - iz_2)\xi_2 = 0\,, \end{cases} \tag{2.14}$$

where it takes the explicit form of an $O(-1) + O(-1)$ bundle over $\mathbb{CP}^1$.[9] Now, the transition (2.10) takes the system of N_c D6 branes on S^3 to an S^2 with N_c units of RR flux [10].

Similarly, we can consider the situation in figure 1a with an O4 plane added on top of the N_c D4 branes. In type IIB, it amounts to performing an orientifold projection that fixes the S^2 so that there is an O5 plane wrapped on S^2. In the mirror picture of type IIA, we must have an O6 plane wrapped on S^3. Thus, one considers the orientifold projection on the deformed conifold [27]:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu\,, \qquad \text{where} \quad (z_i; \bar{z}_i) \equiv (\bar{z}_i; z_i)\,, \tag{2.15}$$

combined with the exchange of the left and right movers on the worldsheet. Adding N_c D6 branes on top of the O6 plane gives rise to an SO(N_c) or Sp($N_c/2$) gauge theory. To identify what the orientifold projection on the deformed conifold becomes after the transition to the resolved conifold, one extends the orientation reversal action to the variables $\xi_{1,2}$. Actually, there is a unique way to do it and the IIA background becomes

$$\begin{cases} (z_1 + iz_2)\xi_1 - (z_3 + iz_4)\xi_2 = 0 \\ (z_3 - iz_4)\xi_1 + (z_1 - iz_2)\xi_2 = 0 \end{cases}, \quad \text{where} \quad (z_i, \xi_1, \xi_2; c.c.) \equiv (\bar{z}_i, -\bar{\xi}_2, \bar{\xi}_1; c.c.)\,. \tag{2.16}$$

Note that a fixed point of the involution would have $\xi_1 = \xi_2 = 0$ which is forbidden. As

[9] In fact, since ξ_1 and ξ_2 must not vanish simultaneously, the determinant of the coefficients of $\xi_{1,2}$ in eq. (2.14) must vanish identically. This determinant is $\sum_i z_i^2$, hence one can replace the first equation in (2.14) by $\sum_i z_i^2 = 0$.

expected, the involution is therefore freely acting since the S^3 has disappeared. Thus, there are neither O6 planes nor D6 branes in this phase. However, there is still a 2-cycle in this background since the base S^2 of the resolved conifold is now replaced by $\mathbb{RP}^2$. By conservation of RR charge, there are also $(N_c/2) \mp 2$ units of flux on $\mathbb{RP}^2$, where $\mp$ is the sign of the O6 plane. Physically, the SO(N_c) or Sp($N_c/2$) group confines [27].

2.2.1 Introducing matter in the type IIA geometry

In type IIB, the implementation of matter and flavor symmetry was done by considering two 2-cycles intersecting at a point with D5 branes wrapped on them. Thus, in type IIA we are looking for another 3-cycle that intersects S^3. In [28], such a geometry was considered.[10] In the deformed conifold of eq. (2.11), an S^3 was identified as the fixed point set of the antiholomorphic involution $z_i \to \bar{z}_i$. Another SLAG is fixed by the involution $z_{1,2} \to \bar{z}_{1,2}$, $z_{3,4} \to -\bar{z}_{3,4}$. It is given by

$$y_{1,2} = x_{3,4} = 0\,, \qquad x_1^2 + x_2^2 = \mu + y_3^2 + y_4^2\,. \tag{2.17}$$

In this equation, $y_{3,4}$ are arbitrary and parametrize a complex plane $\mathbb{C}$. Also, the modulus of $x_1 + ix_2$ is fixed but not its phase. Altogether, eq. (2.17) defines a 3-cycle of topology $\mathbb{C} \times S^1$. Note that the intersection between this cycle and the S^3 of eq. (2.12) satisfies

$$x_{3,4} = y_i = 0\,, \qquad x_1^2 + x_2^2 = \mu\,, \tag{2.18}$$

which is an S^1. If we wrap $N_f - N_c$ D6 branes on the S^3 and N_f D6 branes on $\mathbb{C} \times S^1$, we thus obtain an SU($N_f - N_c$) gauge theory with N_f hypermultiplets of quarks $(q, \tilde{q})$ in the fundamental representation — a "magnetic" theory. Note that for a geometry where $\mathbb{C} \times S^1$ would be replaced by $S^2 \times S^1$ at finite volume, the SU(N_f) flavor symmetry would be gauged. Also, there would be a chiral field M in the adjoint representation of SU(N_f) coupled to the quarks via a classical superpotential $\mathcal{W} = qM\tilde{q}$. However, in our case $M \equiv 0$ since the N_f hypermultiplets are massless.

On the resolved conifold, we already saw that the S^3 fixed by $z_i \to \bar{z}_i$ in $T^*(S^3)$ is replaced by an S^2. To see what $\mathbb{C} \times S^1$ becomes after the conifold transition, one extends to the resolved conifold the antiholomorphic involution that was fixing it: $z_{1,2} \to \bar{z}_{1,2}$, $z_{3,4} \to -\bar{z}_{3,4}$, $\xi_1 \to \bar{\xi}_2$, $\xi_2 \to \bar{\xi}_1$. The new fixed point set satisfies

$$y_{1,2} = x_{3,4} = 0\,, \qquad \lambda \equiv \frac{\xi_1}{\xi_2} = \frac{1}{\bar{\lambda}}\,, \qquad \begin{cases} x_1^2 + x_2^2 = y_3^2 + y_4^2 \\ (iy_3 + y_4)\lambda + (x_1 - ix_2) = 0 \end{cases}\,, \tag{2.19}$$

where λ thus parametrizes an S^1. For any non-vanishing $y_3 + iy_4$, i.e. parametrizing $\mathbb{C}^*$, λ is uniquely determined and the phase of $x_1 + ix_2$ parametrizes an S^1. On the contrary, at the origin $y_3 + iy_4 = 0 = x_1 + ix_2$, it is the phase λ that is arbitrary. Altogether, the topology of this 3-cycle is thus again $\mathbb{C} \times S^1$ and it intersects the S^2 ($z_i = 0$, ξ_1/ξ_2 arbitrary) along its equator ($z_i = 0$, $\xi_1/\xi_2 = \lambda$ arbitrary). Therefore, the N_f D6 branes wrapped on $\mathbb{C} \times S^1$

[10] The aim of [28] was to provide some quantitative check of the pure Yang-Mills duality conjecture of [10, 25].

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remain present on both sides of the transition. However, the $N_f - N_c$ D6 branes wrapped on S^3 in the deformed conifold have disappeared but imply the presence of $N_f - N_c$ units of RR flux on S^2 in the resolved conifold. Thus, in this geometrical phase, the open string sector does not contain massless gauge bosons and quarks any more. This is expected in the IR of the gauge theory we started with on $T^*(S^3)$, due to confinement, where a mass gap is generated.[11]

Since the above duality is an extention of the conjecture of [10] where quarks are included, it would be very interesting to give quantitative checks of it, for example in the spirit of [29].

The generalization to $\mathrm{SO}(N_f - N_c + 4) \times \mathrm{Sp}(N_f/2)$ (N_f even) and $\mathrm{Sp}((N_f - N_c - 4)/2) \times \mathrm{SO}(N_f)$ groups is straightforward. One has to consider the orientifold projection on $T^*(S^3)$ given in eq. (2.15) with $N_f - N_c \pm 4$ D6 branes on top of the O6 plane wrapped on S^3. Also, there are N_f flavor branes wrapped on $\mathbb{C}/\mathbb{Z}_2 \times S^1$, where $\mathbb{Z}_2$ acts as $y_3 + iy_4 \to -y_3 - iy_4$. The singular points on this 3-cycle are precisely the S^1 of the intersection with S^3. After the conifold transition, the geometry is given in eq. (2.16). The O6 plane and $N_f - N_c \pm 4$ D6 branes on S^3 are replaced by $\mathbb{RP}^2$ with $(N_f - N_c/2)$ units of RR flux, while the N_f flavor branes remain.

In section 2.1 we saw that when matter in the fundamental representation is introduced, both from the brane point of view of figures 1–3 and the type IIB geometry, one can also describe electric-magnetic duality. For completeness, we next review how this can also be done in type IIA geometry. In fact, there exists an alternative local construction described in [6], where two phases are naturally related to each other by Seiberg duality, when expected.

Consider the type IIA compactification on a CY, whose local geometry is described by

$$\begin{cases} V^2 + V'^2 = (Z - a_1)(Z - a_2) \\ W^2 + W'^2 = Z - b\,, \end{cases} \tag{2.20}$$

where V, V', W, W', Z are complex variables and a_1, a_2, b are complex parameters. This manifold contains two non-trivial circles S^1_V, S^1_W parametrized by the rotation angles of two $\mathrm{SO}(2,\mathbb{R})$ matrices acting on the column vectors of components V, V' and W, W', respectively. When Z equals a_1 or a_2, S^1_V degenerates. Similarly, when Z equals b, S^1_W degenerates. Thus, by considering the segment $[a_1, a_2]$ in the Z-plane and S^1_V, S^1_W as a fiber on it, one obtains a 3-cycle of topology $S^2 \times S^1$. Similarly, it can be seen that two other 3-cycles of topology S^3 can be associated to the segments $[a_1, b]$ and $[b, a_2]$. We shall denote such a 3-cycle by its base segment in the Z-plane. Let us wrap N_c D6 branes on $[a_1, b]$ and N_f D6 branes on $[b, a_2]$. Requiring an unbroken $\mathcal{N} = 1$ supersymmetry in space-time implies that a_1, b and a_2 are aligned in the Z-plane so that we can choose them to be along the $\mathrm{Re}\, e(Z)$-axis. If we start with a configuration $a_1 < b < a_2$, we thus have branes wrapped on two S^3's. Notice that at the intersection $Z = b$ of their base segments,

[11] Actually, this is expected when $N_f > 3N_c$; in the window $(3/2)N_c < N_f < 3N_c$ one expects an interacting conformal field theory of quarks and gluons [3], while for $N_f \leq (3/2)N_c$ the magnetic theory is IR free. It is not clear to us how to distinguish the various cases from geometry.

S^1_W vanishes but S^1_V is of finite size. Hence the two S^3's intersect along a circle. Thus, the brane system generates an $\mathcal{N}=1$ $\mathrm{SU}(N_c)\times\mathrm{SU}(N_f)$ gauge theory in four dimensions with a massless hypermultiplet $(Q,\tilde{Q})$ in the (N_c,N_f) representation.

The previous CY can now be deformed [6] by changing the complex structure b such that $b<a_1<a_2$. If $N_f>N_c$, the wrapped branes will combine such that we end up with N_f-N_c D6 branes wrapped on the 3-sphere $[b,a_1]$ and N_f others wrapped on the 3-cycle $[a_1,a_2]$. Note that the latter is of topology $S^2\times S^1$ and intersects the 3-sphere along the finite size circle S^1_V that sits at $Z=a_1$. As a result, this system describes an $\mathcal{N}=1$ $\mathrm{SU}(N_f-N_c)\times\mathrm{SU}(N_f)$ gauge theory with a hypermultiplet $(q,\tilde{q})$ in the (N_f-N_c,N_f) representation coupled to a chiral field M in the adjoint of $\mathrm{SU}(N_f)$, with a classical superpotential $\mathcal{W}=qM\tilde{q}$. Thus, the above geometric transition realizes the same electric-magnetic duality we encountered in the brane and type IIB pictures of figures 3 and 6. It can be summarized schematically by the sequence

$$\mathrm{IIA}\quad:\qquad (S^3,S^3)\to(0,\text{singular 3-cycle})\to(\tilde{S}^3,S^2\times S^1)\,, \tag{2.21}$$

where on each side of the transition the 3-cycles intersect along an S^1. Notice that the first of these S^3's can actually be deformed into $\tilde{S}^3$ by considering a path $(b-a_1)\to-(b-a_1)$ in complex structure moduli where the 3-sphere never vanishes. Such a path corresponds in the brane picture of figures 1 and 3 to a motion in the plane (x^6,x^7), hence turning on also a FI D-term in the low energy SYM [4]. This complex structure deformation in type IIA

$$\mathrm{IIA}\quad:\qquad S^3\to 0\to\tilde{S}^3\,, \tag{2.22}$$

is mirror dual to the 2-sphere flop in the type IIB geometry:

$$\mathrm{IIB}\quad:\qquad S^2\to 0\to\tilde{S}^2\,. \tag{2.23}$$

The generalization of the type IIA geometric description of Seiberg duality for SO and Sp groups is realized by implementing an orientifold projection [6].

The link from these electric and magnetic pictures to the brane configuration of figure 3 is found as follows [6]. Each of the equations in (2.20) takes the form of an elliptic fibration over the Z-plane with a monodromy transformation around each point $Z=a_{1,2}$ or $Z=b$, respectively. After two T-dualities on the circles S^1_V and S^1_W to another type IIA description, this is translated into the presence of three Neveu-Schwarz fivebranes. Two of them are parallel and can be identified by convention with NS branes (see eq. (2.1)), while the third one can be identified with an NS' brane. Also, the transverse coordinate x^6+ix^7 can be identified with Z in eq. (2.20). Therefore, the two NS branes sit at $x^6=a_1$ and $x^6=a_2$, while the NS' brane sits at $x^6=b$. Since the 3-cycles in the geometric approach are $S^1_V\times S^1_W$ fibrations over segments in the Z-plane, the D6 branes wrapped on them have Dirichlet boundary conditions in the directions dual to the circles on which we T-dualize. As a result, they give rise to D4 branes stretched between the NS and NS' branes along the direction x^6, as in figure 3.

To conclude this section, we would like to make the link between the geometric descriptions of type IIA and type IIB in section 2.1. The brane picture of section 2.1 is related to the type IIB geometry by a T-duality T_6. It is also related to the type IIA geometry via two T-dualities that act on the phases of the complex planes $x^4 + ix^5$ and $x^8 + ix^9$. As a result, the IIB and IIA geometries are related into each other by three T-dualities, as expected for mirror descriptions of CY threefolds [30].

2.3 Type IIA geometry versus M-theory geometry

Lifting flat parallel type IIA D6 branes to M-theory gives a purely geometrical compactification on a Taub-NUT space [9, 7]. Some cases of D6 branes in curved backgrounds have also been considered [11, 12] and [31]–[37]. In general, a type IIA compactification on a manifold of reduced holonomy with D6 branes wrapped on cycles is lifted in M-theory to a purely gravitational background of different reduced holonomy [31, 32]. More precisely, if one considers the type IIA string on a manifold $\mathcal{M}_d$, the M-theory compact space will have locally the form $\mathcal{M}_d \times S^1$. However, the S^1 is non-trivially fibered over $\mathcal{M}_d$ so that its radius vanishes identically on the submanifolds on which D6 branes are wrapped. Once we are in M-theory, one can ask whether the transitions considered in type IIA with branes would have some simple geometrical interpretations.

We first review the situation for the geometrical transition associated to confinement of pure $\mathcal{N} = 1$ SYM theory [11, 12, 13]. Consider M-theory compactified on Spin(S^3), the spin bundle over S^3 [38]. The metric of this manifold is known, while topologically it looks like the space

$$|u_1|^2 + |u_2|^2 - |u_3|^2 - |u_4|^2 = V\,, \tag{2.24}$$

where u_i ($i = 1, \ldots, 4$) are complex variables and V is a real parameter [12]. Spin(S^3) admits an SU(2)^3 isometry group in which various U(1) subgroups can be chosen to perform a Kaluza-Klein (KK) reduction to a weak coupling type IIA description. An example of such a U(1) acts in the model of eq. (2.24) as

$$U(1) \quad : \quad (u_3, u_4) \to (e^{i\alpha} u_3, e^{i\alpha} u_4)\,. \tag{2.25}$$

For the massless spectrum in M-theory to contain some non-abelian gauge group, one can consider orbifolds of the previous manifold. For instance, we can choose the discrete subgroup of the U(1) isometry generated by

$$\sigma \quad : \quad (u_3, u_4) \to (e^{2i\pi/N_c} u_3, e^{2i\pi/N_c} u_4)\,, \tag{2.26}$$

and focus on Spin(S^3)$/\mathbb{Z}_{N_c}$.

Let us consider first the case $V > 0$. The fixed point set of σ and the U(1) action satisfies

$$u_3 = u_4 = 0\,, \qquad |u_1|^2 + |u_2|^2 = V\,, \tag{2.27}$$

which is the S^3 base of an $\mathbb{R}^4/\mathbb{Z}_{N_c}$ fibration. This describes a pure $\mathcal{N} = 1$ SU(N_c) gauge theory. The dimensional reduction along $S^1 \cong$ U(1) gives a type IIA string theory compactified on a CY threefold with a non-trivial S^3 cycle, $T^*(S^3)$ of eq. (2.11), under the

[12] In fact, eq. (2.24) is just a model for topological properties that are relevant for our purpose. In particular, the metric derived from this equation is *not* of G_2 holonomy and *not* even Ricci flat.

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identification $\mu \equiv V$, together with N_c D6 branes wrapped on S^3 [12]. As explained in section 2.2, two T-dualities translate this system to the type IIA brane configuration of figure 1a.

When $V < 0$, the orbifold is freely acting, as can be seen from the fixed point set (2.27) which is now empty. Therefore, there is no trace anymore of the gauge group, a fact that has been interpreted in [12] as confinement of SU(N_c). Geometrically, there is an S^3 flop transition at $V = 0$, where the 3-sphere (2.27) present for $V > 0$ is replaced by a Lens space S^3/σ

$$u_1 = u_2 = 0\,, \qquad -|u_3|^2 - |u_4|^2 = V\,, \quad \text{where} \quad (u_3, u_4) \equiv e^{2i\pi/N_c}(u_3, u_4)\,, \tag{2.28}$$

for $V < 0$. Schematically, the transition in M-theory takes the form:

$$\text{M-theory} \quad : \qquad S^3 \to 0 \to S^3/\sigma\,. \tag{2.29}$$

The Lens space can be seen as a Hopf fibration over S^2, whose fiber is precisely U(1). Upon KK reduction along $S^1 \cong$ U(1), it gives rise to the S^2 of the resolved conifold with N_c units of RR flux through it. This is again consistent with the interpretation of confinement, as reviewed in section 2.2.

Similarly, we can consider singularities in M-theory in the D-series. For $N_c \geq 8$ and even, one thus defines Spin(S^3)$/\tau$, where τ has two generators [27]

$$\tau \quad : \quad (u_3, u_4) \to (e^{2i\pi/(N_c-4)}u_3, e^{2i\pi/(N_c-4)}u_4) \quad \text{and} \quad (u_3, u_4) \to (\bar{u}_4, -\bar{u}_3)\,. \tag{2.30}$$

This definition of τ is equivalent to the dihedral group $D_{N_c/2}$, which implies that physically there is an SO(N_c) gauge group for $V > 0$ that confines for $V < 0$. When $V > 0$, the dimensional reduction along $S^1 \cong$ U(1) gives $N_c/2$ D6 branes, an O6 plane and $N_c/2$ mirror branes wrapped on the S^3 of $T^*(S^3)$ in type IIA. This is precisely the orientifold model of eq. (2.15). When $V < 0$, the first generator of τ implies that the reduction of the Lens space S^3/τ gives an S^2 with $N_c - 4$ units of RR flux. Identifying the complex coordinate of this S^2 with $-u_3/u_4$, the second generator of τ implies the modding action $-u_3/u_4 \to \bar{u}_4/\bar{u}_3$. This background is precisely the orientifold of the resolved conifold in eq. (2.16) under the identification $\xi_1 = -u_3$, $\xi_2 = u_4$.

Actually, the classical type IIA description on the deformed conifold for $\mu \equiv V \gg 0$ provides a good approximation of the physics only in the UV, where the non-abelian gauge group is weakly coupled. When $\mu \equiv V$ is still positive but decreases, the SYM coupling increases and the naive classical description becomes less accurate. When we pass to negative values of V in M-theory, the situation gets even worse. In general, in the type IIA reduction it is necessary to compute worldsheet instanton contributions in order to describe strong coupling effects in the corresponding gauge theory, like confinement. As an example, when Spin(S^3)$/\sigma$ is reduced along $S^1 \cong$ U(1), the type IIA worldsheet instantons are computed by closed string topological amplitudes. In [34], Spin(S^3)$/\sigma$ is instead reduced along another $S^1 \cong$ U(1)$''$ (we shall define later in eq. (2.40)). For $V > 0$, the type IIA background is in this case $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_{N_c}$, thus describing an explicit SU(N_c) gauge symmetry in space-time.[13] This classical background is accurate to describe the SYM physics in the UV. However, for $V < 0$, the classical type IIA background is still $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_{N_c}$,[14] which cannot be trusted for describing accurately the SU(N_c) gauge theory in the IR. This is due to the presence of large instanton corrections arising from open string worldsheets with disk topology. To compute them, one can map the system in this phase to a dual one, namely a type IIB string theory on the mirror CY with a D5 brane wrapped on a curve.

We have reviewed that for even $N_c \geq 8$ the lift to M-theory of a type IIA orientifold on $T^*(S^3)$ with an O6 plane and a total of N_c D6 branes wrapped on S^3 generates an SO(N_c) gauge group.[15] In the following we shall discuss the case $N_c = 4$ that will help us for treating models in later sections. In the notation of eq. (2.11) and defining x^{10} as a coordinate along a circle S^1 of radius R_{10}, let us consider

$$\frac{T^*(S^3) \times S^1}{w\mathcal{I}}\,, \quad \text{where} \quad w: \; z_i \to \bar{z}_i \quad \text{and} \quad \mathcal{I}: \; x^{10} \to -x^{10}\,. \tag{2.31}$$

For $\mu > 0$, this orbifold is topologically $S^3 \times (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as an inversion on each coordinate of $\mathbb{R}^3$ and x^{10}. Therefore, there are two copies of S^3 of A_1 singularities, one at $x^{10} = 0$ and the other at $x^{10} = \pi R_{10}$, generating an SU(2)^2 gauge symmetry. The metric of the double cover is simply

$$ds^2 = ds^2_{CY} + \frac{R_{10}^2}{l_{\rm p}^2}\left(dx^{10}\right)^2\,, \tag{2.32}$$

where $ds^2_{\rm CY}$ is the CY metric and $l_{\rm p}$ is the eleven-dimensional Planck mass. Thus, identifying R_{10} with the type IIA coupling and taking the limit $R_{10} \to 0$, the geometric background becomes $T^*(S^3)/w$, i.e. an orientifold of type IIA with an O6 plane wrapped on S^3. Since these remarks apply as well for compact CY's, where the total RR charge must cancel, there must be two D6 branes (and their mirrors) on top of the O6 plane. Altogether, the branes and orientifold generate an SO(4) $\cong$ SU(2)^2 gauge group as in M-theory [39].

At $\mu = 0$, $T^*(S^3)$ used in the construction of the orbifold has become a cone and we first choose to desingularize it by blowing up an S^2. Therefore, from the 7-dimensional point of view, we are now considering M-theory on

$$\frac{\text{resolved conifold} \times S^1}{w'\mathcal{I}}\,, \tag{2.33}$$

where, in the notations of eq. (2.14), the involution $w'\mathcal{I}$ acts as

$$w': \; \begin{cases} z_i \to \bar{z}_i \\ \xi_1 \to -\bar{\xi}_2\,, \quad \xi_2 \to \bar{\xi}_1 \end{cases} \qquad \text{and} \qquad \mathcal{I}: \; x^{10} \to -x^{10}\,. \tag{2.34}$$

Since w' is freely acting, there is no obvious trace of the non-abelian gauge group in M-theory, a fact that we are again going to interpret as confinement. Actually, the two S^3's

[13]In addition, there is a D6 brane wrapped on a SLAG of topology $\mathbb{C}/\mathbb{Z}_{N_c} \times S^1$.

[14]With a D6 brane wrapped on a SLAG of topology $\mathbb{C} \times S^1$.

[15]Actually, the M-theory backgrounds based on orbifolds of Spin(S^3) are the lifts of type IIA string theories in the infinite string coupling limit. For finite string coupling, one has to replace Spin(S^3) by a G_2 holonomy manifold constructed in [35].

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of A_1 singularities present in the space (2.31) for $\mu > 0$ have become $(S^2 \times S^1)/w'\mathcal{I} \cong \mathbb{RP}^3$ in the manifold (2.33).[16] As explained near eq. (2.29), such flop transitions $S^3 \to 0 \to \mathbb{RP}^3$ characterize confinement of each SU(2) factor. Taking the limit $R_{10} \to 0$ in eq. (2.32) sends us again to an orientifold of type IIA on (resolved conifold)$/w'$, where there are no fixed points i.e. no orientifold plane. By conservation of charge, there are no D6 branes and thus no gauge group, as in M-theory. Actually, this background is precisely the one given in eq. (2.16) that has been shown to describe the confining phase of SO(4): An orientifold of the resolved conifold with no RR flux on $\mathbb{RP}^2$ since $(N_c/2) - 2 = 0$. Thus the reduction $R_{10} \to 0$ of the M-theory backgrounds (2.31) and (2.33) is similar to the reduction along $S^1 \cong \mathrm{U}(1)$ of $\mathrm{Spin}(S^3)/\tau$ for the case of an SO(4) gauge theory.

However, when $\mu = 0$ in eq. (2.31), we can alternatively desingularize the orbifold by passing into the phase $\mu < 0$ of $(T^*(S^3) \times S^1)/w\mathcal{I}$.[17] In that case, w is freely acting and there is no obvious sign of the non-abelian gauge group present for $\mu > 0$. Again, there are flop transitions $S^3 \to 0 \to S^3/w \cong \mathbb{RP}^3$ in M-theory[18] and this third phase should also describe confinement of $\mathrm{SU}(2)^2$.[19] Sending $R_{10} \to 0$ in eq. (2.32), one obtains a description in terms of an orientifold of type IIA on $T^*(S^3)/w$, with no fixed points i.e. no orientifold plane. As before, conservation of RR charge implies that there are no D6 branes and thus no gauge group, as in M-theory. This orientifold background is very different from the one considered in the previous paragraph that describes accurately confinement of SO(4) once closed string worldsheet instanton corrections are taken into account. Locally, when passing from $\mu > 0$ to $\mu < 0$ in the orientifold of type IIA on $T^*(S^3)/w$, a transition $S^3 \to 0 \to S^3/w \cong \mathbb{RP}^3$ occurs instead of $S^3 \to 0 \to \mathbb{RP}^2$ we had before. In the phase $\mu < 0$, one then has to compute corrections, eventually by considering a dual description relevant for calculations.

2.3.1 A new duality in type IIA

The duality of eq. (2.10), where branes on S^3 and flux on S^2 are interchanged, is more easily understood when lifted to M-theory [12]. Starting again from M-theory on $\mathrm{Spin}(S^3)/\sigma$, we are going to consider now a KK reduction along a different M-theory circle and obtain a new duality conjecture in type IIA.

Let us define another subgroup of isometry that acts as

$$U(1)' \quad : \quad (u_1, u_2) \to (e^{i\beta} u_1, e^{i\beta} u_2)\,, \tag{2.35}$$

whose fixed point set is given in eq. (2.28). For $V < 0$, this set is a Lens space S^3/σ and actually the whole 7-manifold satisfies $\mathrm{Spin}(S^3)/\sigma = \mathrm{Spin}(S^3/\sigma) \cong \mathbb{R}^4 \times S^3/\sigma$, where

[16] Since $(S^2 \times S^1)/w'\mathcal{I}$ is closed with $\pi_1\left((S^2 \times S^1)/w'\mathcal{I}\right) = \mathbb{Z}_2$, $(S^2 \times S^1)/w'\mathcal{I} \cong \mathbb{RP}^3$ by Thurston elliptization conjecture.

[17] Due to the action of the antiholomorphic involution w on $T^*(S^3)$, we have now to distinguish the cases $\mu > 0$ and $\mu < 0$.

[18] S^3 for $\mu > 0$ and S^3/w for $\mu < 0$ are parametrized by $\mathrm{Re}\, e(z_i)$ and $\Im m(z_i)$ in eq. (2.11), respectively.

[19] As will be seen in section 4, when one considers the similar phase diagram where an arbitrary CY replaces $T^*(S^3)$ in the orbifold (2.31), the two phases where SO(4) confines are distinguished by the number of neutral chiral multiplets.

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U(1)$'$ acts only on $\mathbb{R}^4$. Thus, upon KK reduction along $S^1 \cong$ U(1)$'$, the type IIA geometry becomes $T^*(S^3/\sigma) \cong \mathbb{R}^3 \times S^3/\sigma$, where a D6 brane is wrapped on the base S^3/σ. An equation for S^3/σ is (2.12), where μ is now identified with $-V > 0$ and a $\mathbb{Z}_{N_c}$ modding action on x_i is understood. The manifold $T^*(S^3/\sigma)$ is thus the deformed conifold of eq. (2.11) on which the modding action on $\mathrm{Re}\, e(z_i)$ is lifted to z_i by holomorphicity:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = -V \,, \qquad \text{where} \quad \left\{ \begin{array}{l} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \begin{pmatrix} \cos(\omega) & \text{-}\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \equiv \begin{pmatrix} \cos(\omega) & \text{-}\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \end{array} \right. \,, \tag{2.36}$$

with $\omega = 2\pi/N_c$. Clearly, we thus have $T^*(S^3/\sigma) = T^*(S^3)/\sigma$ where we use the same symbol σ to refer to an orbifold action on the whole manifold or restricted to S^3. To make contact with the notation of eq. (2.24), the relations between the coordinates of the Lens spaces in $\mathrm{Spin}(S^3)/\sigma$ and $T^*(S^3/\sigma)$ are $u_3 = x_1 + ix_2$, $u_4 = x_3 + ix_4$. In this phase, the massless spectrum in M-theory and type IIA contains a chiral field whose scalar is $\Upsilon \equiv C + i\mu/N_c$, where C is the expectation value of the 3-form on S^3/σ whose volume is $-V \equiv \mu$. In type IIA, there is also an $\mathcal{N} = 1$ U(1) vector multiplet on the worldvolume of the brane.

For $V > 0$, $\mathrm{Spin}(S^3)/\sigma \cong \mathbb{R}^4/\mathbb{Z}_{N_c} \times S^3$, where the 3-sphere is defined in eq. (2.27). Since on this S^3, U(1)$'$ acts freely, the 3-sphere is a Hopf fibration over an S^2, whose fiber is $S^1 \cong$ U(1)$'$. Therefore, S^3 is reduced in type IIA to an S^2 with one unit of RR two-form flux on it, while the $\mathbb{R}^4/\mathbb{Z}_{N_c}$ fibration gives rise to an A_{N_c-1} singularity over the S^2. Physically, the flux on S^2 is interpreted as a magnetic FI term [40, 41] that breaks $\mathcal{N} = 2 \to \mathcal{N} = 1$ in string theory [42] (see also [43, 44] for supersymmetry breaking in string theory). To be more precise concerning the geometry, since the KK reduction of $\mathrm{Spin}(S^3)$ along $S^1 \cong$ U(1)$'$ is the resolved conifold, the reduction of $\mathrm{Spin}(S^3)/\sigma$ gives in type IIA the resolved conifold in which the $\mathbb{R}^4$ fiber over S^2 is modded by $\mathbb{Z}_{N_c}$. Since in eq. (2.14) (ξ_1, ξ_2) parametrizes $\mathbb{CP}^1 \cong S^2$, while the z_i's constrained by the equations parametrize $\mathbb{R}^4$, the type IIA background takes the form:

$$\left\{ \begin{array}{l} (z_1 + iz_2)\xi_1 - (z_3 + iz_4)\xi_2 = 0 \\ (z_3 - iz_4)\xi_1 + (z_1 - iz_2)\xi_2 = 0 \end{array} \right. \,, \quad \text{where} \quad \left\{ \begin{array}{l} z_1 \pm iz_2 \equiv e^{\pm i\omega}(z_1 \pm iz_2) \\ z_3 \pm iz_4 \equiv e^{\pm i\omega}(z_3 \pm iz_4) \end{array} \right. \,. \tag{2.37}$$

In this phase, the massless spectrum in M-theory and type IIA contains a non-perturbative SU(N_c) gauge group. The theta angle and gauge coupling are given in type IIA by the complexified Kähler modulus of S^2: $T \equiv B + i\,\mathrm{vol}(S^2)$, where B is the NS-NS 2-form flux on S^2. In M-theory variables, $T \equiv C + iV$, where C is the flux of the 3-form on S^3 whose volume is V. Note that in type IIA, the dimensional reduction of the RR 3-form on S^2 gives rise to an additional perturbative $\mathcal{N} = 1$ vector multiplet. Combined with the non-abelian factor, the string theory gauge group is thus U(N_c).

In M-theory, the two phases $V < 0$ and $V > 0$ are dual to each other in the sense that one passes from one to the other by a change of energy scale from the IR ($V < 0$) to the UV ($V > 0$) of the corresponding gauge theory. Therefore, the type IIA descriptions to which they descend upon dimensional reduction should also be dual in that sense. Note that this is consistent with the fact that the two backgrounds of eqs. (2.36) and (2.37) are related by a conifold transition since the modding actions on the z_i's are equivalent. Hence, we are led to conjecture:

In type IIA, a $\mathbb{CP}^1$ of A_{N_c-1} singularity with one unit of RR flux through it is dual to a Lens space $S^3/\mathbb{Z}_{N_c}$ with one D6-brane wrapped on it. They describe the UV and IR physics of a pure $\mathcal{N}=1$ $\mathrm{SU}(N_c)$ gauge theory, respectively.

In fact, the $\mathrm{SU}(N_c)$ gauge group explicit in the UV confines in the IR, while the diagonal U(1) remains spectator through the transition. As in the large N duality conjecture of [10], Υ should be interpreted physically as the lowest component of the gaugino condensate superfield, $g_s \mathrm{Tr} W^\alpha W_\alpha = \Upsilon + \cdots$, where W^α is the supersymmetric field strength of the $\mathrm{SU}(N_c)$ factor. In particular, it would be very interesting to rederive in the context of the above duality conjecture the relation between T and Υ:

$$(e^{\Upsilon} - 1)^N = e^{-T} \,. \tag{2.38}$$

The generalization of the duality conjecture to the $\mathrm{SO}(N_c)$ groups (for even $N_c \geq 8$) is straightforward. Consider M-theory on $\mathrm{Spin}(S^3)/\tau$. For $V < 0$, the reduction along $S^1 \cong \mathrm{U}(1)'$ gives the manifold $T^*(S^3/\tau)$, where S^3/τ is the Lens space associated to the dihedral group, and a D6 brane is wrapped on S^3/τ. The action on $u_{3,4}$ of the two generators of τ in eq. (2.30) are translated to an action on x_i in eq. (2.12) due to the identification $u_3 = x_1 + ix_2$, $u_4 = x_4 + ix_5$. Then, holomorphicity implies that $T^*(S^3/\tau) = T^*(S^3)/\tau$ in this phase. The defining equation of this manifold is given in eq. (2.36), where $\omega = 2\pi/(N_c - 4)$, and a second identification

$$(z_1, z_2, z_3, z_4) \equiv (z_3, -z_4, -z_1, z_2) \tag{2.39}$$

is understood. For $V > 0$, the reduction gives an S^2 with one unit of RR flux and a $D_{N_c/2}$ singular fibration over it. Globally, this space is given in eq. (2.37) with the additional identification (2.39). Finally, the previous considerations can also be applied to the $E_{6,7,8}$ groups.

2.3.2 Introducing flavor in the M-theory geometry?

We have seen how two phases associated to the UV and IR physics in a pure SYM theory realized in type IIA can be lifted to M-theory. In section 2.2, confinement in presence of massless quarks multiplets have also been considered in type IIA string theory. Our aim is now to propose an M-theory description that could realize such a SYM theory coupled to quarks.

With this in mind, we first consider another isometry subgroup considered in [34, 13], $\mathrm{U}(1)''$, that corresponds in the model of eq. (2.24) to the action

$$U(1)'' \quad : \quad (u_2, u_3) \to (e^{i\gamma} u_2, e^{i\gamma} u_3) \,. \tag{2.40}$$

If we consider the discrete subgroup of $\mathrm{U}(1)''$ generated by

$$\rho \quad : \quad (u_2, u_3) \to (e^{2i\pi/N_f} u_2, e^{2i\pi/N_f} u_3) \,, \tag{2.41}$$

we can construct the orbifold $\mathrm{Spin}(S^3)/\mathbb{Z}_{N_f}$. The fixed point set of ρ and $\mathrm{U}(1)''$ is

$$u_2 = u_3 = 0\,, \qquad |u_1|^2 - |u_4|^2 = V\,, \tag{2.42}$$

whose topology is $\mathbb{C} \times S^1$ and is the base of an $\mathbb{R}^4/\mathbb{Z}_{N_f}$ fibration.[20] If $\mathbb{C}$ was replaced by a finite volume $\mathbb{CP}^1$, since $b_1(\mathbb{CP}^1 \times S^1) = 1$, this would generate an $\mathrm{SU}(N_f)$ gauge theory with one chiral field in the adjoint representation. However, since we deal with the infinite volume base $\mathbb{C} \times S^1$, these fields are frozen and we are left with an $\mathrm{SU}(N_f)$ global symmetry. From the results of [34, 13], the dimensional reduction of $\mathrm{Spin}(S^3)/\rho$ along $S^1 \cong \mathrm{U}(1)''$ gives a type IIA string theory on the space $\mathbb{C}^3$, where N_f coincident D6 branes are wrapped on a SLAG of topology $\mathbb{C} \times S^1$, thus producing an $\mathrm{SU}(N_f)$ global symmetry in type IIA.

For integers $N_f > N_c$, let us denote by $\tilde{\sigma}$ the generator similar to σ, where N_c is replaced by $N_f - N_c$:

$$\tilde{\sigma} \quad : \quad (u_3, u_4) \to \left(e^{2i\pi/(N_f - N_c)} u_3, e^{2i\pi/(N_f - N_c)} u_4\right). \tag{2.43}$$

Since $\tilde{\sigma}$ and ρ commute, the group they generate is of finite order. We shall denote it by $\{\tilde{\sigma}, \rho\}$. We now consider the space $\mathrm{Spin}(S^3)/\{\tilde{\sigma}, \rho\}$ as a good candidate for describing a "magnetic" $\mathcal{N} = 1$ $\mathrm{SU}(N_f - N_c)$ gauge theory with N_f flavors of quarks.

For $V > 0$, the singular point set of $\tilde{\sigma}$ is subject to a quotient by ρ implying an identification $u_2 \equiv e^{2i\pi/N_f} u_2$ in eq. (2.27). As a result, the 3-space $S^3/\mathbb{Z}_{N_f}$ has singular points:

$$u_2 = u_3 = u_4 = 0\,, \qquad |u_1|^2 = V\,, \tag{2.44}$$

which is an S^1. Similarly, the singular point set of ρ is subject to a quotient by $\tilde{\sigma}$, implying an identification $u_4 \equiv e^{2i\pi/(N_f - N_c)} u_4$ in eq. (2.42). Therefore, the resulting fixed point set $\mathbb{C}/\mathbb{Z}_{N_f - N_c} \times S^1$ is singular along the S^1 given in (2.44). Notice that the group element $\tilde{\sigma}\rho$ does not introduce new singular points since its fixed points are also given by (2.44). To summerize, the orbifold points of $\mathrm{Spin}(S^3)/\{\tilde{\sigma}, \rho\}$ consist of two singular 3-spaces $S^3/\mathbb{Z}_{N_f}$ and $\mathbb{C}/\mathbb{Z}_{N_f - N_c} \times S^1$ glued together precisely along their S^1 of singularities, as depicted in figure 7.

From an M-theory point of view, it is not yet known how to determine the massless spectrum of this background. Actually, there should be gauged $\mathrm{SU}(N_f - N_c)$ and global $\mathrm{SU}(N_f)$ symmetries arising from the $\mathbb{R}^4/\mathbb{Z}_{N_f - N_c}$ and $\mathbb{R}^4/\mathbb{Z}_{N_f}$ fibrations over the bases $S^3/\mathbb{Z}_{N_f}$ and $\mathbb{C}/\mathbb{Z}_{N_f - N_c} \times S^1$, respectively. In addition, there could be massless matter localized at the intersection S^1 of the bases, in the bifundamental representation of $\mathrm{SU}(N_f - N_c) \times \mathrm{SU}(N_f)$, that could be interpreted as N_f flavors of massless quarks $(q, \tilde{q})$. In that case, this geometry would be a good candidate for describing the lift to M-theory of the "magnetic" theory described in section 2.2: in type IIA on $T^*(S^3)$, $N_f - N_c$ and N_f D6 branes are wrapped on S^3 and $\mathbb{C} \times S^1$ that are intersecting along S^1. However, we have not been able to find a KK reduction of M-theory on $\mathrm{Spin}(S^3)/\{\tilde{\sigma}, \rho\}$ that would send us to this type IIA picture.

[20]For $V > 0$, $\mathbb{C} \times S^1$ is parametrized by u_4 and the phase of u_1, while for $V < 0$, the roles of u_4 and u_1 is reversed.

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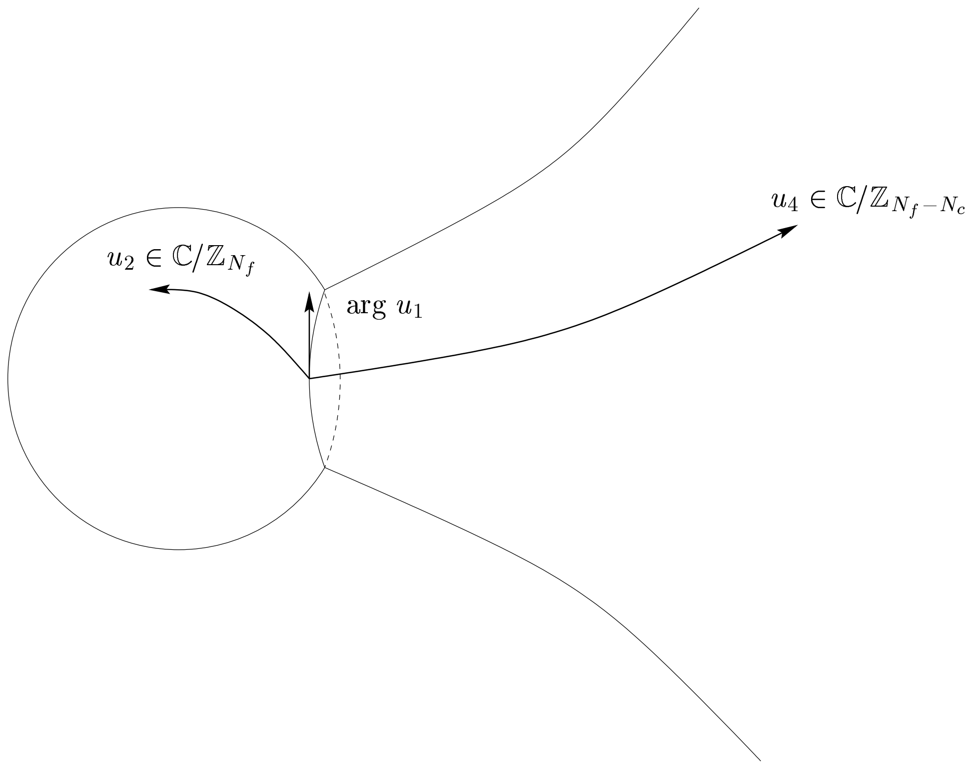

Figure 7: An $S^3/\mathbb{Z}_{N_f}$ of $A_{N_f-N_c-1}$ singularities and a $\mathbb{C}/\mathbb{Z}_{N_f-N_c} \times S^1$ of A_{N_f-1} singularities. The two bases intersect along an S^1.

For $V < 0$, on one hand $\tilde{\sigma}$ and $\tilde{\sigma}\rho$ are freely acting. On the over hand, the fixed point set of ρ is given by eq. (2.42) with the identification $u_4 \equiv e^{2i\pi/(N_f-N_c)}u_4$, giving rise to a topology $\mathbb{C} \times S^1/Z_{N_f-N_c} \cong \mathbb{C} \times S^1$. Hence, we are left with an $\mathrm{SU}(N_f)$ global symmetry in M-theory associated with the $\mathbb{R}^4/\mathbb{Z}_{N_f}$ fibration over $\mathbb{C}\times S^1$. Again, this could be interpreted as the IR physics of the confining $\mathrm{SU}(N_f - N_c)$ gauge theory with matter. Upon KK reduction, this would be consistent with the type IIA configuration described after eq. (2.19): N_f D6 branes wrapped on $\mathbb{C} \times S^1$ in the resolved conifold with $N_f - N_c$ units of RR flux through the S^2.

As a final remark, M-theory on $\mathrm{Spin}(S^3)/\{\tilde{\sigma}, \rho\}$ can be mapped to various type IIA models that could shed some light on the physics described by this geometry. As an example, for $V > 0$, the KK reduction on $S^1 \cong \mathrm{U}(1)''$ results in a type IIA string theory on a background $\mathbb{C}^2/\mathbb{Z}_{N_f-N_c} \times \mathbb{C}$. In addition, there are N_f coincident D6 branes wrapped on a SLAG $\mathbb{C}/\mathbb{Z}_{N_f-N_c} \times S^1$. This situation is treated in [34] for a single wrapped D-brane.

3. Connecting/disconnecting and vanishing 3-spheres

In this section we consider a compact M-theory background where various transitions take place such as the connection of two disjoint 3-spheres of singularities. We first describe the M-theory construction and determine the massless spectrum in each phase. The models we shall consider are similar to those studied in [20]. Then we shall give quantitative and physical interpretations of the transitions by mapping them to dual descriptions involving some local geometries or brane configurations considered in the previous sections.

3.1 The M-theory geometrical setup

We start by considering a two-parameter sub-set of CY's $\mathcal{C}_1$ in the family $\mathbb{CP}^4_{11222}[8]$. The Hodge numbers of the threefolds are $h_{11} = 2$ and $h_{12} = 86$. The defining polynomial

$$p_1 \equiv z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \psi z_2^4 z_6^2)^2 + z_4^4 + z_5^4 = 0\,, \tag{3.1}$$

is written in terms of two complex parameters ψ, ϕ and the projective coordinates subject to two scaling actions $\mathbb{C}^*$, whose weights are given in the following table:

$$\begin{array}{c|cccccc} & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \hline \mathbb{C}^*_1 & 0 & 0 & 1 & 1 & 1 & 1 \\ \mathbb{C}^*_2 & 1 & 1 & 0 & 0 & 0 & -2 \end{array} \tag{3.2}$$

The presence of the variable z_6 together with the second scaling action is due to the blow up of the $\mathbb{Z}_2$ singularity sitting at $z_1 = z_2 = 0$ in the ambient $\mathbb{CP}^4_{11222}$: In the resulting toric space, there are excluded sets

$$(z_1, z_2) \neq (0,0) \quad \text{and} \quad (z_3, z_4, z_5, z_6) \neq (0,0,0,0)\,. \tag{3.3}$$

Notice that we have chosen to work in a slice of the complex structure moduli space by fixing to zero most of the coefficients of the monomials allowed by the $\mathbb{C}^*$ actions.

A CY in the family $\mathcal{C}_1$ is singular when the equation (3.1) is non-transverse, i.e. when $p_1 = 0$, $dp_1 = 0$. This happens only in charts where z_2 and z_6 do not vanish so that we can rescale them to 1. Then, the singularities occur

$$\begin{array}{llcll} \text{for any } \psi, & \text{at } \phi = +1 & : & (i^k, 1, \pm\sqrt{\psi}, 0, 0, 1) & ,(k = 0, \ldots, 3)\,, \\ & \text{or } \phi = -1 & : & (i^k e^{i\pi/4}, 1, \pm\sqrt{\psi}, 0, 0, 1) & ,(k = 0, \ldots, 3)\,, \\ \text{and for any } \phi, & \text{at } \psi = \pm\sqrt{\phi^2 - 1} & : & (\phi^{1/4} i^k, 1, 0, 0, 0, 1) & ,(k = 0, \ldots, 3)\,. \end{array} \tag{3.4}$$

It happens that the determinant of second derivatives $\det(\partial_A \partial_B p_1)$ $(A, B = 1, 3, 4, 5)$ has two vanishing eigenvalues at these points. Therefore, these isolated singularities are not nodal points. We shall come back to this remark later. Since the maps $(\phi, z_1) \to (-\phi, e^{i\pi/4} z_1)$ and $(\psi, z_6) \to (-\psi, i z_6)$ leave p_1 invariant, the threefolds associated to $(\pm\phi, \pm\psi)$ are one and only one. Thus, from the CY point of view, we could consider complex parameters ψ and ϕ such that $\mathrm{Re}\, e(\phi), \mathrm{Re}\, e(\psi) \geq 0$ only.

However, we are interested in $\mathcal{N} = 1$ compactifications of M-theory on orbifolds of the form similar to eq. (2.31):

$$\mathcal{G}_1 = \frac{\mathcal{C}_1 \times S^1}{w\mathcal{I}}\,, \tag{3.5}$$

where the involution $w\mathcal{I}$ acts simultaneously on the CY and S^1 as:

$$w:\; z_i \to \bar{z}_i \quad (i = 1, \ldots, 6)\,, \qquad \mathcal{I}:\; x^{10} \to -x^{10}\,. \tag{3.6}$$

Actually, for w to be a symmetry of $\mathcal{C}_1$, we restrict now ϕ and ψ to real values. From the point of view of $\mathcal{G}_1$, since $z_1 \to e^{i\pi/4} z_1$ and $z_6 \to i z_6$ do not commute with w, ϕ and $-\phi$ as well as ψ and $-\psi$ are no longer equivalent.

We now determine the fixed point set of the orbifold. Since these points satisfy $(z_i, \bar{z}_i, x^{10}) \equiv (\bar{z}_i, z_i, -x^{10})$, they are described by two copies of the special lagrangian 3-cycle Σ in $\mathcal{C}_1$ fixed by w. One copy sits at $x^{10} = 0$, while the second sits at $x^{10} = \pi R_{10}$.[21] Defining x_i and y_i to be the real and imaginary parts of z_i, Σ is given by $y_i = 0$ and eq. (3.1) for real unknowns x_i:

$$x_6^4(x_1^8 + x_2^8 - 2\phi x_1^4 x_2^4) + (x_3^2 - \psi x_2^4 x_6^2)^2 + x_4^4 + x_5^4 = 0\,. \tag{3.7}$$

Note that x_6 cannot vanish in this equation, since otherwise it would imply $x_{3,4,5} = 0$ as well, which is forbidden (see eq. (3.3)). Thus, we can rescale x_6 to 1. Similarly, x_2 can always be rescaled to 1 since $x_2 = 0$ would also imply $x_1 = 0$. In eq. (3.7), the remaining unknowns are not projective anymore and we can solve for x_1^4:

$$x_1^4 = \phi \pm \sqrt{\phi^2 - \left[1 + (x_3^2 - \psi)^2 + x_4^4 + x_5^4\right]}\,. \tag{3.8}$$

Clearly, for $\phi < 1$, there is no real solution for x_1^4 and Σ is therefore empty. For $\phi = 1$, there are solutions for $x_3^2 = \psi$, $x_4 = x_5 = 0$, $x_1^4 = 1$, when $\psi \geq 0$ and Σ consists of four points

$$\Sigma = \{(\pm 1, 1, \pm\sqrt{\psi}, 0, 0, 1)\}\,. \tag{3.9}$$

At these points, the CY is also singular, as can be seen from eq. (3.4).

The situation for $\phi > 1$ is more delicate. By defining the variables

$$X_1 = x_1^4 - \phi\,, \qquad X_3 = x_3^2 - \psi\,, \qquad X_j = x_j^2 \,\mathrm{sign}(x_j)\,, \quad (j = 4, 5)\,, \tag{3.10}$$

the equation for Σ turns out to be

$$X_1^2 + X_3^2 + X_4^2 + X_5^2 = \phi^2 - 1\,, \tag{3.11}$$

which describes an S^3 of radius $\sqrt{\phi^2 - 1}$. However, whereas $X_{4,5}$ and $x_{4,5}$ are in one-to-one correspondence, the map from x_1 to X_1 is two-to-one as can be seen from the relation $-\sqrt{\phi^2 - 1} \leq x_1^4 - \phi \leq \sqrt{\phi^2 - 1}$ that gives two disconnected sets of solutions

$$\begin{aligned} &0 < (\phi - \sqrt{\phi^2 - 1})^{1/4} \leq x_1 \leq (\phi + \sqrt{\phi^2 - 1})^{1/4} \\ \text{and} \quad &-(\phi + \sqrt{\phi^2 - 1})^{1/4} \leq x_1 \leq -(\phi - \sqrt{\phi^2 - 1})^{1/4} < 0\,. \end{aligned} \tag{3.12}$$

Thus, the total fixed point set Σ consists of the disjoint union of two isomorphic sets Σ_+ and Σ_-, where x_1 is always strictly positive in the former and strictly negative in the

[21] In general, Σ itself is a union of disconnected 3-cycles. When the first Betti number b_1 of one of these components is non-vanishing, it is conjectured [17] that it is possible to blow up the singularities lying on it (if some other condition is also satisfied). In that case, the holonomy of the resulting space is G_2. When b_1 of each component vanishes, none of them can be desingularized and the holonomy is the semi-product of SU(3) and $\mathbb{Z}_2$. However, the four-dimensional physics of M-theory on these spaces is in both cases $\mathcal{N} = 1$ and we shall refer to them as G_2 orbifolds.

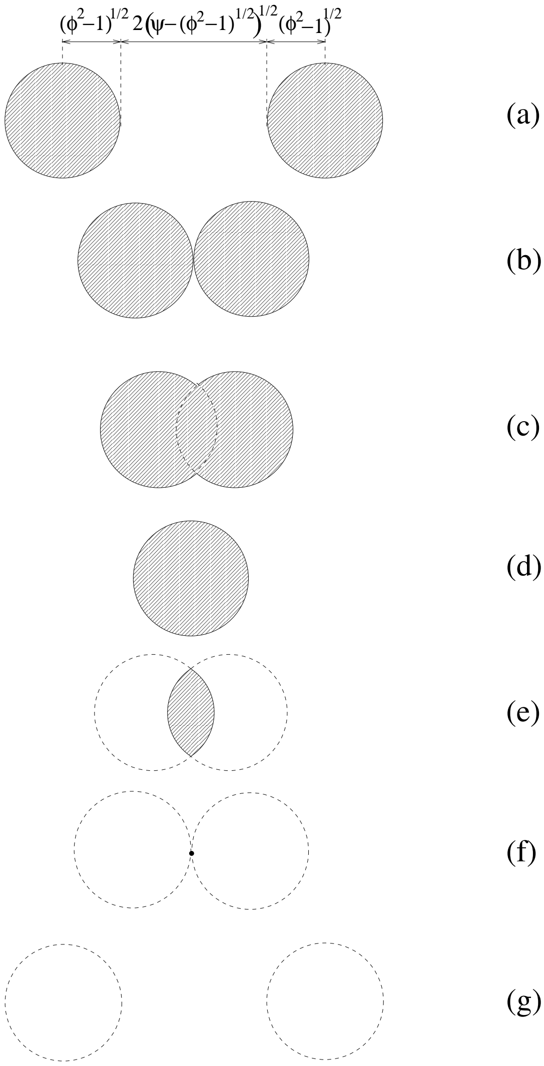

Figure 8: (a) Σ_+ is composed of two 3-spheres that apprach each other. (b) They intersect at a singular point of the manifold. (c,d,e) They are connected. Topologically, it is equivalent to a single S^3. (f) The S^3 shrinks to a singular point of the manifold. (g) Σ_+ is empty.

latter. We shall concentrate now on Σ_+, keeping in mind that one obtains Σ_- from Σ_+ by changing $x_1 \to -x_1$. Finally, we have to translate the fixed point coordinate X_3 into x_3. From the inequalities $-\sqrt{\phi^2-1} \leq x_3^2 - \psi \leq \sqrt{\phi^2-1}$, the following discussion arises, which is illustrated by figure 8 that represents the projection $x_4 = x_5 = 0$ of Σ_+ in the plane (x^1, x^3):

- For $\psi > \sqrt{\phi^2-1}$: There exist two disjoint sets of solutions for x_3

$$\begin{aligned} 0 < (\psi - \sqrt{\phi^2-1})^{1/2} \leq x_3 \leq (\psi + \sqrt{\phi^2-1})^{1/2} \\ \text{or} \quad -(\psi + \sqrt{\phi^2-1})^{1/2} \leq x_3 \leq -(\psi - \sqrt{\phi^2-1})^{1/2} < 0 \,, \end{aligned} \tag{3.13}$$

so that Σ_+ is composed of two disconnected 3-spheres (see figure 8a):

$$\text{for } \psi > \sqrt{\phi^2-1} \quad : \qquad \Sigma_+ = S^3 \cup S^3 \,. \tag{3.14}$$

• For $\psi = \sqrt{\phi^2 - 1}$: The two previous S^3's intersect at one point $(\phi^{1/4}, 1, 0, 0, 0, 1)$, so that Σ_+ is singular at this point (see figure 8*b*). Note that the CY is also singular at the same point (see eq. (3.4)).

• For $-\sqrt{\phi^2 - 1} < \psi < \sqrt{\phi^2 - 1}$: In this phase, the set of solutions for x_3 takes the form

$$\begin{aligned} & 0 \leq x_3 \leq (\psi + \sqrt{\phi^2 - 1})^{1/2} \\ \text{or} \quad & -(\psi + \sqrt{\phi^2 - 1})^{1/2} \leq x_3 \leq 0 \,, \end{aligned} \tag{3.15}$$

so that

$$\text{for } -\sqrt{\phi^2 - 1} < \psi < \sqrt{\phi^2 - 1} \quad : \quad \Sigma_+ = S^3 \,, \tag{3.16}$$

where this S^3 is actually the connected sum of the two 3-spheres we had before (see figures 8*c*,*d*,*e*).

• For $\psi = -\sqrt{\phi^2 - 1}$: The size of the previous S^3 has vanished and we have a singular 3-cycle (see figure 8*f*) :

$$\text{for } \psi = -\sqrt{\phi^2 - 1} \quad : \quad \Sigma_+ = \{(\phi^{1/4}, 1, 0, 0, 0, 1)\} \,, \tag{3.17}$$

which again corresponds to a singular point in the CY (see eq. (3.4)).

• Finally, for $\psi < -\sqrt{\phi^2 - 1}$: There is no solution for x_3 and we have (see figure 8*g*):

$$\text{for } \psi < -\sqrt{\phi^2 - 1} \quad : \quad \Sigma_+ = \emptyset \,. \tag{3.18}$$

To summarize, at fixed $\phi > 1$ and according to the values of ψ, the fixed point set Σ_+ is:

$$\begin{array}{lll} \psi > \sqrt{\phi^2 - 1}, & \Sigma_+ = S^3 \cup S^3 & \text{(disconnected union)}\,, \\ \psi = \sqrt{\phi^2 - 1}, & \Sigma_+ = S^3 \cup S^3 & \text{(intersecting at one singular point)}\,, \\ |\psi| < \sqrt{\phi^2 - 1}, & \Sigma_+ = S^3 \# S^3 \cong S^3 & \text{(connected sum of the } S^3\text{'s)}\,, \\ \psi = -\sqrt{\phi^2 - 1}, & \Sigma_+ = \{(\phi^{1/4}, 1, 0, 0, 0, 1)\} & \text{(one singular point)}\,, \\ \psi < -\sqrt{\phi^2 - 1}, & \Sigma_+ = \emptyset & \text{(no fixed point)}\,. \end{array} \tag{3.19}$$

This determines three phases in the (ϕ, ψ) plane drawn in figure 9, where the topology of Σ_+ is represented. We now describe the massless spectrum in each phase.

– **Spectrum in phase I:** $\phi < 1$ **or** $\phi \geq 1$, $\psi < -\sqrt{\phi^2 - 1}$**.** In this phase, the compact space is smooth and the four-dimensional massless spectrum is obtained by dimensional reduction of the eleven-dimensional supergravity multiplet [45]. Besides the $\mathcal{N} = 1$ gravity multiplet, the resulting spectrum contains b_2 vector multiplets and b_3 neutral chiral multiplets,[22] where b_2 and b_3 are the Betti numbers of $\mathcal{G}_1$. From our construction of $\mathcal{G}_1$, $b_{2,3}$

[22] These multiplets are in fact linear multiplets due to a Peccei-Quinn symmetry remanent of the gauge invariance $C \to C + d\Lambda_2$ of the eleven dimensional supergravity 3-form.

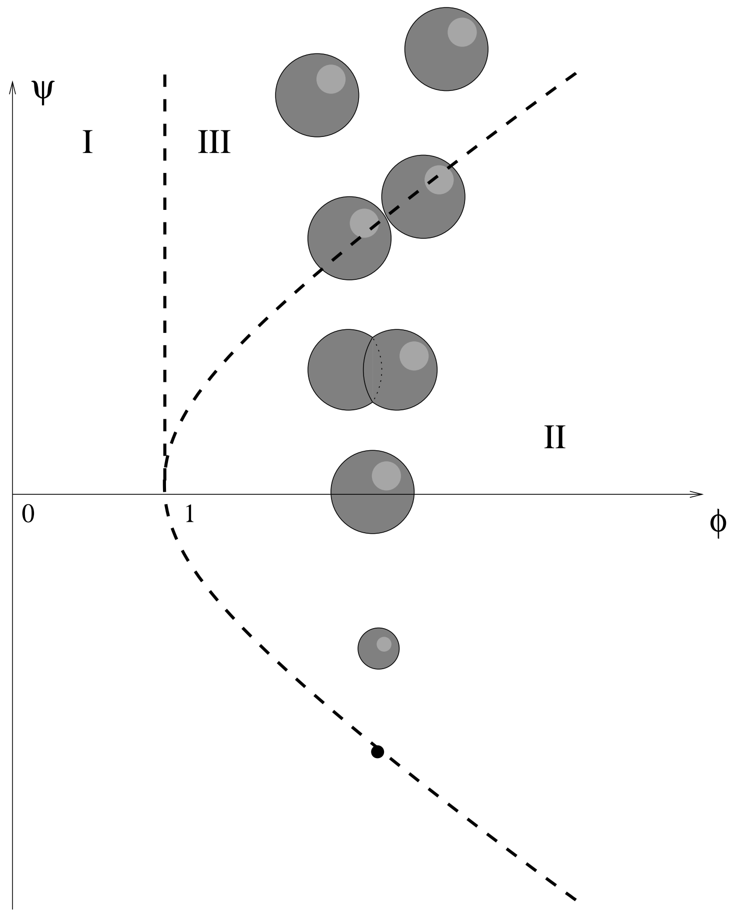

Figure 9: Phase diagram of the model based on $\mathcal{G}_1$. Σ_+ is represented for various values of (ϕ, ψ). When non-empty, it is composed of 3-spheres. The transitions are physical and should occur at finite distance in moduli space.

can be expressed in terms of h_{11} and h_{12}, the Hodge numbers of $\mathcal{C}_1$. Actually, a 2-cycle on the orbifold arises from a 2-cycle in $\mathcal{C}_1$ which is even under w. Let us define the number of even (odd) 2-cycles in $\mathcal{C}_1$ to be h_{11}^+ (h_{11}^-). On the other hand, the product of S^1 by any of the h_{11}^- odd 2-cycles of $\mathcal{C}_1$ gives rise to a 3-cycle on $\mathcal{G}_1$. In addition, the 3-cycles even under w in $\mathcal{C}_1$ remain in the orbifold. Noticing that there are as many even as odd 3-cycles in $\mathcal{C}_1$, one obtains

$$b_2 = h_{11}^+ \,, \qquad b_3 = \frac{h_{30} + h_{03}}{2} + \frac{h_{21} + h_{12}}{2} + h_{11}^- = 1 + h_{12} + h_{11}^- \,. \tag{3.20}$$

In the present case, there are $h_{11} = 2$ cohomology classes on the CY. The first one being the pullback on $\mathcal{C}_1$ of the Kähler form of $\mathbb{CP}^1_{11222}$, it is odd under w. The second one is Poincaré dual of the holomorphic blow up $\mathbb{CP}^1$ at $(z_1, z_2) = (0, 0)$, which is also odd under w. As a result, we have $h_{11}^+ = 0$ and the Betti numbers of $\mathcal{G}_1$ are

$$b_2 = 0 \qquad \text{and} \qquad b_3 = 1 + 86 + 2 = 89 \,. \tag{3.21}$$

Thus, we have

$$\text{89 chiral multiplets} \qquad \text{and} \qquad \text{no gauge group}\,. \tag{3.22}$$

– **Spectrum in phase II:** $\phi > 1$, $|\psi| < \sqrt{\phi^2 - 1}$. In this phase, the massless spectrum still contains the states arising from the reduction from eleven to four dimensions of the supergravity multiplet. This gives the $\mathcal{N} = 1$ gravity multiplet together with 89 chiral multiplets. In addition [11, 20], there are states localized on the fixed point set $(\Sigma_+ \cup \Sigma_-) \times \{0, \pi R_{10}\}$, which is composed of four disconnected copies of $S^3 \# S^3 \cong S^3$. At any point of one of these 3-spheres, the geometry looks like $\mathbb{R}^4/\mathbb{Z}_2 \times S^3$, where S^3 is parametrized by the x_i's, while $\mathbb{R}^4/\mathbb{Z}_2$ accounts for the y_i's and x^{10}. Thus, there is an SU(2) gauge group arising from M-theory on $\mathbb{R}^4/\mathbb{Z}_2$ further compactified to four dimensions on S^3. Since $b_1(S^3) = 0$, this results in an $\mathcal{N} = 1$ vector multiplet of SU(2) with no adjoint matter for each S^3. Including the spectrum arising from the four disjoint 3-spheres in $\Sigma \times \{0, \pi R_{10}\}$, we obtain

$$\text{1 vector multiplet of } \mathrm{SU}(2)^4 \qquad \text{and} \qquad \text{89 neutral chiral multiplets}\,. \tag{3.23}$$

Note that since $b_1(S^3) = 0$, it is not possible to desingularize the A_1 singularities to obtain a smooth G_2-holonomy manifold.

– **Spectrum in phase III:** $\phi > 1$, $\psi > \sqrt{\phi^2 - 1}$. This case is treated as the previous one. The only difference is that each S^3 in phase II is replaced by a disjoint union $S^3 \cup S^3$ in phase III. Therefore the massless spectrum is

$$\text{1 vector multiplet of } \mathrm{SU}(2)^8 \qquad \text{and} \qquad \text{89 neutral chiral multiplets}\,. \tag{3.24}$$

3.2 Brane and field theory interpretation of the geometrical phases

We have seen that there are three geometrical phases in figure 9. Two transitions, II→I and III→I, correspond to passing from the UV to the IR of the SYM theories that confine. This was explained in a non-compact model in [12, 27], reviewed in section 2.3, and generalized to compact G_2 manifolds of the form $(CY \times S^1)/w\mathcal{I}$ in [20]. We now have to understand the third transition, III→II. We do not know how to determine the physics it describes from a pure M-theoretic point of view, due to additional massless matter that may occur at the transition. Therefore, we shall map the local geometries involved in each transition in M-theory to local configurations of branes in type IIA which are similar to those considered in section 2.1.

We first reduce the M-theory background in phase II to a type IIA orientifold on $\mathcal{C}_1/w$ by identifying the size R_{10} of S^1 with the string coupling. This is similar to what we did for the space of eq. (2.31). In $\mathcal{G}_1$, the set $\Sigma_+ \times \{0, \pi R_{10}\}$ is composed of two 3-spheres that descend in type IIA to a single S^3 on which two D6 branes on top of an orientifold sixplane O6 (as well as two mirror branes) are wrapped [39].[23] Two T-dualities of the form

[23] We only consider the vicinity of the component Σ_+ of Σ; the discussion for the second one Σ_- is independent and similar.

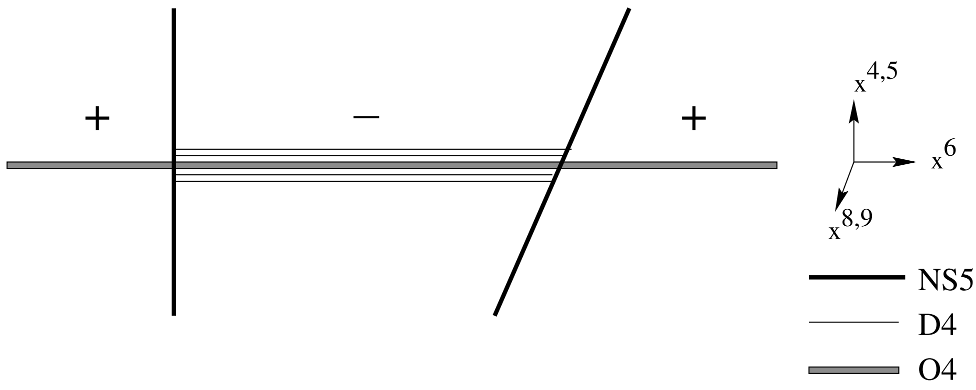

Figure 10: Brane realization of SO(4) SYM.

described in section 2.2 take this, locally, to the type IIA brane configuration in figure 10: NS and NS' branes (see eq. (2.1)) are separated along an O4 plane in the direction x^6. The orientifold has a negative RR charge between the fivebranes and positive charge on the other sides (see [1] for a review). Two D4 branes and their mirror images are stretched between the NS and NS' branes. The low energy gauge theory is indeed a four dimensional $\mathcal{N}=1$ SYM with gauge group $\mathrm{SO}(4) \cong \mathrm{SU}(2)^2$, as in the M-theory description.

On the line $\psi = -\sqrt{\phi^2-1}$ in figure 9, the NS and NS' branes intersect at a point as in figure 11*d*. After the transition from phase II to phase I, the gauge group SO(4) has disappeared. As explained in section 2.1, in the brane picture, this is obtained dynamically by the bending of the fivebranes away from the orientifold, as shown in figure 11*e*. In M-theory, as in the non-compact models considered in section 2.3, the transition should correspond for any of the four 3-spheres of $(\Sigma_+ \cup \Sigma_-) \times \{0, \pi R_{10}\}$ to a flop $S^3 \to 0 \to \mathbb{RP}^3$. Since this transition in M-theory occurs by varying ψ, which is the real part of a complex structure modulus of the underlying CY $\mathcal{C}_1$, $\mathbb{RP}^3$ is the lift to the orbifold $\mathcal{G}_1$ of the 3-cycle of topology $S^3/w \cong \mathbb{RP}^3$ in $\mathcal{C}_1/w$ that exists for $\psi < -\sqrt{\phi^2-1}$ and vanishes at the transition. As an example, for the particular transition $(\psi = 0, \phi > 1) \to (\psi = 0, \phi < 1)$ along the $\psi \equiv 0$ axis of figure 9, the "flopped" 3-sphere of $\mathcal{C}_1$ can be seen as the fixed point locus of an antiholomorphic involution of the form $z_1 \to \bar{z}_2$, $z_2 \to \bar{z}_1$, $z_j \to \bar{z}_j$ $(j = 3,4,5,6)$.[24] At any other point of the transition II→I, the "flopped" 3-sphere in $\mathcal{C}_1$ is a generic special lagrangian 3-cycle and cannot be seen as the fixed point locus of an antiholomorphic involution that acts globally on the CY. Finally, we postpone to section 4 the discussion of the transition II→I from the point of view of the type IIA orientifold on $\mathcal{C}_1/w$.

Next, we consider the transition III→II. We start by reducing the M-theory geometry in phase III of figure 9 to a type IIA orientifold description. Again focusing only on $\Sigma_+ \times \{0, \pi R_{10}\}$ in $\mathcal{G}_1$, for $\psi \gg \sqrt{\phi^2-1}$, locally, we should get two systems in ten dimensions, each of which being T-dual to the brane configuration in figure 10. Since the two systems are connected at the transition $\psi = \sqrt{\phi^2-1}$ (see figure 9), their dual brane configuration should be located on the same orientifold, as is shown in figure 11*a*.

[24] Up to phases appearing in the definition of the involution [20].

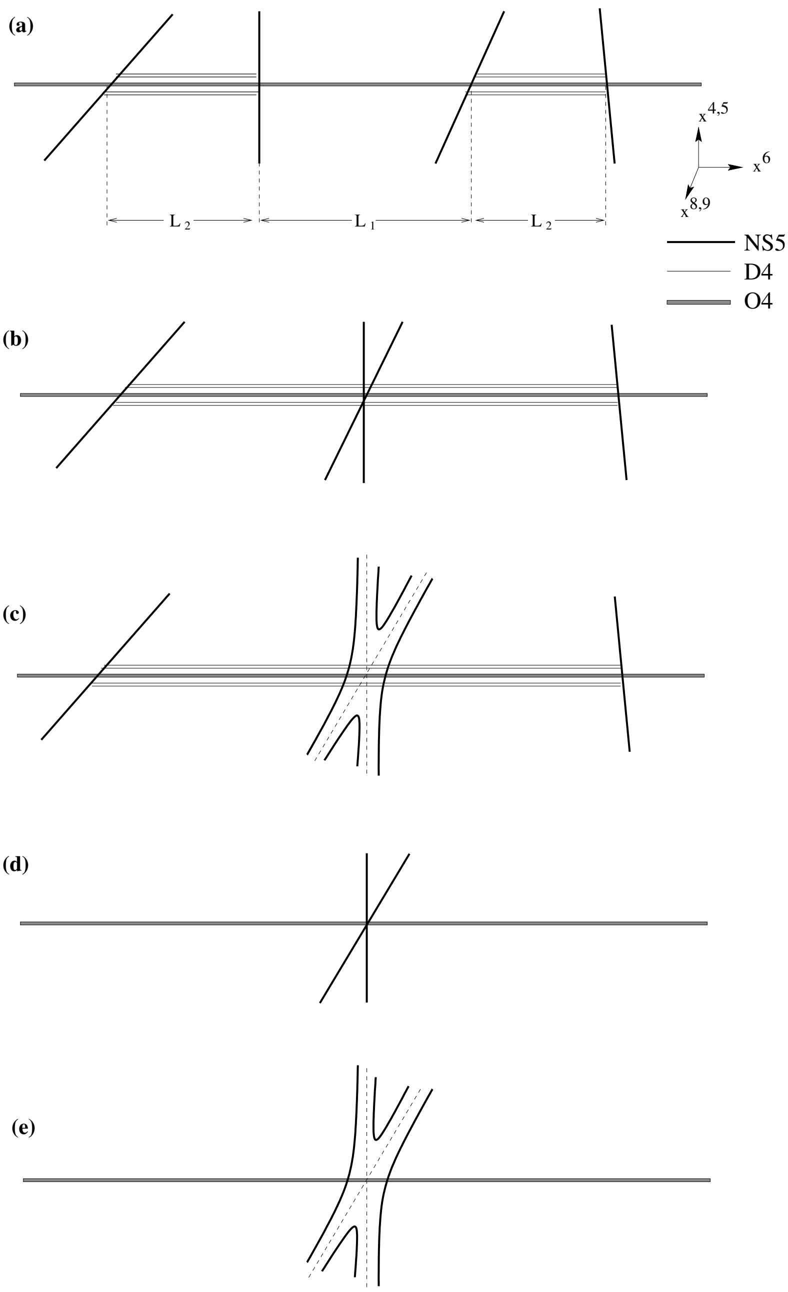

Figure 11: (a) Two SO(4) systems of branes approach each other. (b) The middle branes intersect. (c) Then, they bend. (d) The left over Neveu-Schwarz branes approach each other till they intersect. (e) Finally, they bend.

As ψ approaches $\sqrt{\phi^2 - 1}$, these two systems approach each other. More precisely, the quantity $\psi - \sqrt{\phi^2 - 1}$ is associated with the distance L_1 between the two systems in figure 11a, while $\sqrt{\phi^2 - 1}$ corresponds to the distance L_2 between the NS and NS' in each of the two systems (see also figure 8a for the geometric description). The relative orientation between the various Neveu-Schwarz fivebranes varies as we change ψ and ϕ. In the limit $\psi \to \sqrt{\phi^2 - 1}$, however, the two middle fivebranes are not parallel.[25] Moreover, the other two fivebranes are not mutually parallel with any of the other fivebranes, as shown in figure 11b.

The low energy gauge theory is an $\mathcal{N} = 1$ SO(4) $\times$ SO(4) SYM with a $Q \in (4, 4)$ chiral multiplet. In phase III i.e. figure 11a, Q is massive since it corresponds to the open strings stretched from the D4 branes of one system to those of the other. However, since the G_2 orbifold in M-theory is compact, the mass parameter of Q is actually a field M whose VEV, classically, is a flat direction. This is described in field theory by a tree level superpotential

$$\mathcal{W}_{\text{tree}} = MX \,, \tag{3.25}$$

where

$$X = Q^2 \tag{3.26}$$

is the gauge singlet built out of $Q \in (4, 4)$.[26] Crossing the line $\psi = \sqrt{\phi^2 - 1}$ from phase III to phase II in figure 9, there is a breaking of SO(4) $\times$ SO(4) to a single SO(4). In the brane picture, this is realized dynamically by the bending of the middle fivebranes away from the orientifold, as shown in figure 11c. In the low energy SYM, this quantum effect is taking us away from the origin of the classical moduli space by giving a VEV to the bifundamental Q, thus going to the Higgs branch where SO(4) $\times$ SO(4) is broken to the diagonal SO(4). Since this branch does not exist in the perturbative superpotential of eq. (3.25), the Higgs mechanism that occurs is a non-perturbative effect. Next we shall see that it is possible to describe it in gauge theory.

A dynamical superpotential for X is generated[27] and has two physically distinct phases described by:

$$\mathcal{W}_{\text{dyn},\pm}(X) = \frac{X^2}{\Lambda_\pm} \,, \qquad \Lambda_\pm = \frac{\Lambda_1 \pm \Lambda_2}{2} \,, \tag{3.27}$$

where Λ_1 and Λ_2 are the "QCD scales" of the SO(4) $\times$ SO(4), respectively. In our case, the M-theory geometry is such that the gauge couplings of the two SO(4) factors are equal:

$$\Lambda_1 = \Lambda_2 \equiv \Lambda \,, \tag{3.28}$$

hence[28]

$$\Lambda_+ = \Lambda \,, \qquad \Lambda_- = 0 \,. \tag{3.29}$$

[25] This is due to the existence of a 3-sphere "between the two systems" which shrinks to 0 size in the limit $\psi \to \sqrt{\phi^2 - 1}$. In fact, there are other S^3's that collapse simultaneously to the same point implying that the resulting singularity is more severe than a standard nodal point (we recall that the determinant of second derivative vanishes at this point). All these S^3's are related under the map $(\psi, z_6) \to (-\psi, iz_6)$ to the 3-spheres present in phase I that vanish at $\psi = -\sqrt{\phi^2 - 1}$.

[26] More precisely, there is also a massive adjoint chiral multiplet (Φ_L, Φ_R) of SO(4) $\times$ SO(4) coupled to Q, which upon integrating it out may generate a non-trivial superpotential for X (see [46] for the details in the SU(N_L) $\times$ SU(N_R) case); it is plausible that, due to the symmetric nature of the M-theory background which we consider, such terms will vanish in the corresponding SYM (the analog of the case $N_L = N_R$ and $\mu_L = -\mu_R$ in [46]).

[27] This can be shown, for instance, by "integrating in" [47] Q to pure $\mathcal{N} = 1$ SO(4) $\times$ SO(4) SYM with $\mathcal{W}_{\text{down},\pm} \propto (\Lambda^3_{1,\text{down}} \pm \Lambda^3_{2,\text{down}})$.

[28] In the case $\Lambda_- = 0$ there is no dynamical superpotential for X: $\mathcal{W}_{\text{dyn},-}(X, \Lambda_- = 0) = 0$, since $\mathcal{W}_{\text{down},-} = 0$. Alternatively, this can be understood as the limit $\Lambda_- \to 0$ in eq. (3.27) which sets the constraint $X = 0$.

To obtain the phase structure which we see in the M-theory geometrical picture we assume that there is also a dynamically generated term for M:[29]

$$\mathcal{W}_{\mathrm{dyn},\pm}(M) = \frac{\Lambda_\pm M^2}{4}\,. \tag{3.30}$$

Altogether, the quantum superpotential we expect is:[30]

$$\mathcal{W}_\pm(X,M) = \frac{X^2}{\Lambda_\pm} + MX + \frac{\Lambda_\pm M^2}{4}\,. \tag{3.31}$$

Let us see now that $\mathcal{W}_\pm$ have the required properties.

Varying $\mathcal{W}_\pm$ with respect to X and M give rise to the same equation of motion:

$$X = -\frac{1}{2}\Lambda_\pm M\,. \tag{3.32}$$

Hence the quantum moduli space is composed of two branches $\mathcal{M}_\pm$ which intersect at $M = X = 0$. In the case (3.29), the branch $\mathcal{M}_+$ consists of:

$$\mathcal{M}_+ \quad : \qquad X = -\frac{1}{2}\Lambda_+ M\,. \tag{3.33}$$

This is the Higgs branch associated to the geometrical phase II in figure 9. On the other hand, the branch $\mathcal{M}_-$ consists of:

$$\mathcal{M}_- \quad : \qquad X = 0\,, \qquad M \quad \text{anything}\,. \tag{3.34}$$

We associate this branch with phase III.

Finally, the geometrical phase transition from III to I in figure 9 is similar to the transition from II to I. In the dual brane picture, phase III is described by the system of figure 11a and the transition consists in approaching, intersecting and then bending the NS and NS' of each system of branes separately but simultaneously.

To summerize, a phase diagram with the transitions in the brane configurations is shown in figure 12. All these transitions are smooth in the brane picture and should occur at finite distance in moduli space in the M-theory geometry. In figure 12, they are represented by dashed lines.

4. Type II orientifolds, confinement and Higgs mechanism

Three examples of phase transitions II$\to$I, III$\to$I and III$\to$II have been considered in section 3. In each case, from the M-theory point of view, 3-spheres experience flop transitions, while from the brane point of view, this is realized by the bending of Neveu-Schwarz fivebranes. The transitions were interpreted in field theory as confinement (or quantum

[29] This is the *only* term in M and Λ which is consistent with the symmetries; it is similar to the quantum potential generated for the "meson" in a "magnetic" $\mathcal{N}=1$ theory [3]. Also, the 1/4 normalization of this term is required for the total superpotential to admit a vacuum.

[30] Note that $\mathcal{W}$ is self-dual under the strong-weak coupling duality $\Lambda/2\mu \leftrightarrow 2\mu/\Lambda$ together with $X/\mu^2 \leftrightarrow M/\mu$, where μ is some mass scale.

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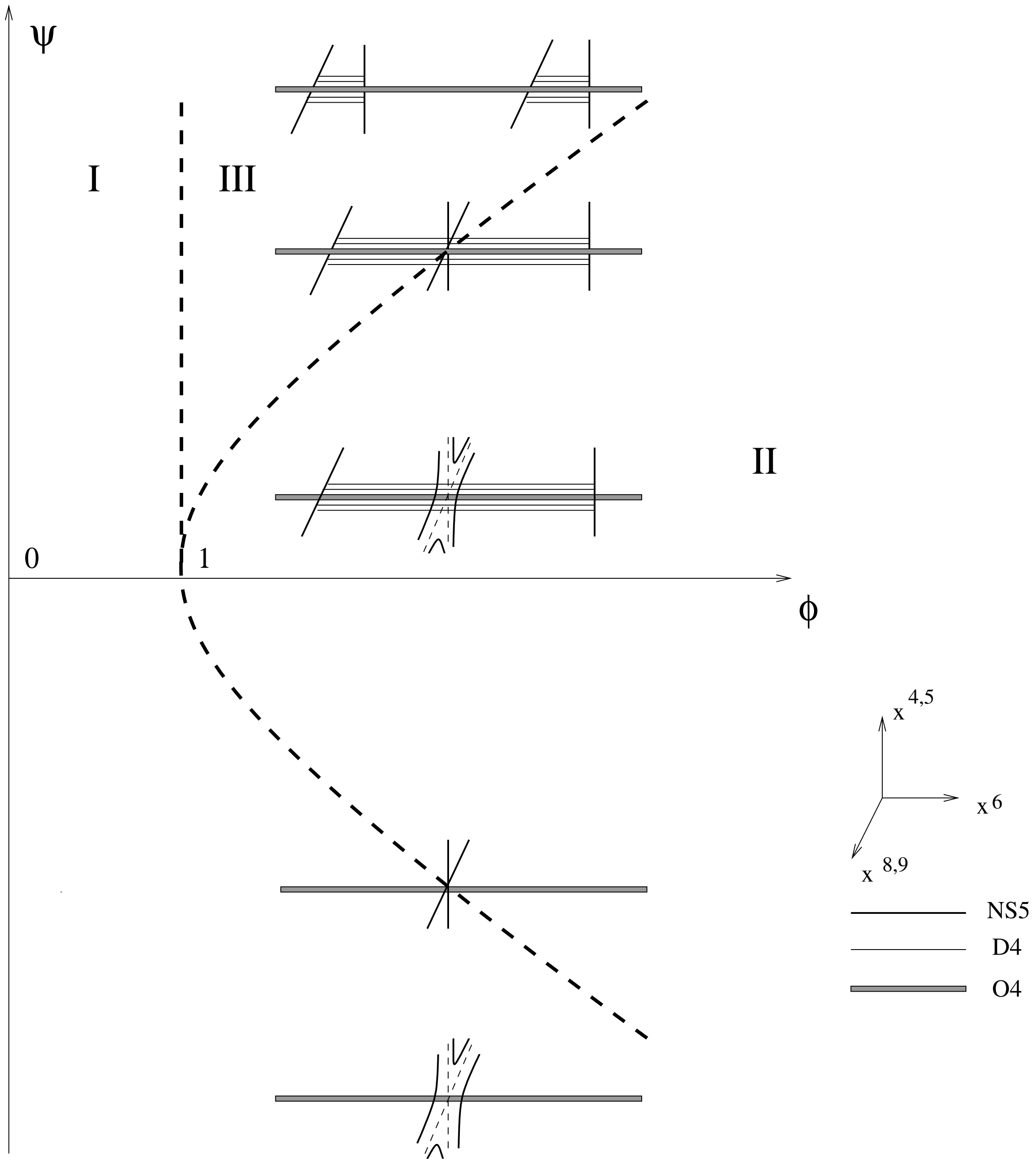

Figure 12: Phase diagram of the model based on $\mathcal{G}_1$. The brane system dual to Σ_+ is represented for various values of (ϕ, ψ).

Higgs effect) of SO(4) gauge groups. To clarify some points from the type IIA orientifold descriptions, let us relate these effects to the transitions that occur in the local model $(T^*(S^3) \times S^1)/w\mathcal{I}$ defined in eq. (2.31).

In the non-compact G_2 orbifold $(T^*(S^3) \times S^1)/w\mathcal{I}$, we saw in section 2.3 that one can perform a conifold transition on the underlying CY threefold $T^*(S^3)$. Thus, one passes into the moduli space of $\left((\text{resolved conifold}) \times S^1\right)/w'\mathcal{I}$ that is related to confinement. In section 4.1, we shall consider the similar transition in the compact case. We shall see that in general the change in the Hodge numbers of the underlying CY has important physical consequences. In addition to confinement, a change of branch in the scalar potential of chiral multiplets takes place.

However, we note that these CY conifold transitions do not occur in the specific model we studied in section 3. This is due to the fact that for $\mathcal{C}_1$, at the local singularities of eq. (3.4), the determinant of second derivatives vanishes. Instead, the transitions for the compact model of section 3 correspond in the local model $(T^*(S^3)\times S^1)/w\mathcal{I}$ to passing from positive values of μ defined in eq. (2.11) to negative values of μ. As explained in section 2.3, when $\mu < 0$, quantum corrections in the classical type IIA orientifold obtained by sending $R_{10} \to 0$ have to be taken into account in order to describe accurately confinement. As an example, one can consider a dual description to compute these corrections. At the end of section 4.1, we shall comment on the situation where the non-compact CY threefold $T^*(S^3)$ is replaced by a compact one.

Finally, the case of a non-abelian Higgs mechanism together with a change of branch in the scalar potential will be considered in section 4.2. This will be treated in the spirit of [23].

4.1 Confinement, scalar potential and conifold transitions

To describe a conifold transition in the underlying CY of a G_2 orbifold, we consider the second model treated in [20],

$$\mathcal{G}_2 = (\mathcal{C}_2 \times S^1)/w\mathcal{I}\,, \tag{4.1}$$

where $\mathcal{C}_2$ is another sub-family of CY threefolds in $\mathbb{CP}^4_{11222}[8]$ defined by

$$p_2 \equiv z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \phi z_6^2 z_2^4)^2 + (z_4^2 - \phi z_6^2 z_2^4)^2 + (z_5^2 - \phi z_6^2 z_2^4)^2 = 0\,. \tag{4.2}$$

The antiholomorphic involution w acts again as $z_i \to \bar{z}_i$, while ϕ is a real parameter. At the particular values $\phi = \pm 1$, $\mathcal{C}_2$ becomes a conifold, whose nodal points[31] are

$$\begin{array}{lll} \text{for } \phi = +1 & : & (i^k, 1, \pm 1, \pm 1, \pm 1, 1) \qquad\quad , (k = 0, \ldots, 3)\,, \\ \text{for } \phi = -1 & : & (i^k e^{i\pi/4}, 1, \pm 1, \pm 1, \pm 1, 1)\ , (k = 0, \ldots, 3)\,, \end{array} \tag{4.3}$$

where the $\pm$ signs are all independent. The phase diagram of the fixed point set $\Sigma \times \{0, \pi R_{10}\}$ of this model is then obtained as in [20, section 3.1]:

$$\begin{array}{lll} \phi > 1\,, & \Sigma = \bigcup_{n=1}^{16} S^3 & (\text{disconnected and of radii} \sqrt{\phi^2 - 1})\,, \\ \phi = 1\,, & \Sigma = \{(\pm 1, 1, \pm 1, \pm 1, \pm 1, 1)\} & (\text{i.e. 16 points})\,, \\ \phi < 1\,, & \Sigma = \emptyset & (\text{no fixed points})\,. \end{array} \tag{4.4}$$

From eq. (4.3), notice that at $\phi = 1$, the 16 vanishing 3-spheres coincide with singular points of $\mathcal{C}_2$, where we shall consider a conifold transition.

In M-theory, as in section 3.1, the 3-spheres of A_1 singularities for $\phi > 1$ give rise to an $SU(2)^{32}$ gauge group. In addition, from eq. (3.20), there are $\mathcal{N} = 1$ vector and chiral

[31] The determinant of second derivatives does not vanish at these singularities.

multiplets in M-theory. Sending $R_{10} \to 0$, one obtains a type IIA orientifold on $\mathcal{C}_2/w$, with two D6 branes (and their mirror partners) and an O6 plane wrapped on each of the 16 S^3's of Σ. There is thus an $\mathrm{SO}(4)^{16}$ gauge group as in M-theory. Also, it is shown in [48] that the massless spectrum arising from invariant forms on an orientifold of CY in type IIA is consisting of [32]

$$h_{11}^{+} \ \text{ vector multiplets} \quad \text{ and } \quad 1 + h_{12} + h_{11}^{-} \ \text{ neutral chiral multiplets} \,. \tag{4.5}$$

Hence, the spectra in the type IIA orientifold and in M-theory coincide. As in the case based on $\mathcal{C}_1$, we have $h_{11}^{+} = 0$ and we have in total

$$1 \text{ vector multiplet of } \mathrm{SO}(4)^{16} \qquad \text{and} \qquad 89 \text{ neutral chiral multiplets} \,. \tag{4.6}$$

At $\phi = 1$, we now choose to blow up an S^2 at each node of the conifold $\mathcal{C}_2$. The CY threefold $\mathcal{C}_2'$ obtained this way can be written as

$$\begin{cases} \left[z_6^2(z_1^4 - z_2^4) + i(z_3^2 - z_6^2 z_2^4)\right] \xi_1 + \left[(z_4^2 - z_6^2 z_2^4) - i(z_5^2 - z_6^2 z_2^4)\right] \xi_2 = 0 \\ -\left[(z_4^2 - z_6^2 z_2^4) + i(z_5^2 - z_6^2 z_2^4)\right] \xi_1 + \left[z_6^2(z_1^4 - z_2^4) - i(z_3^2 - z_6^2 z_2^4)\right] \xi_2 = 0 \end{cases}, \tag{4.7}$$

where $\xi_{1,2}$ are projective coordinates parametrizing $\mathbb{CP}^1 \cong S^2$ as in eq. (2.14). The Hodge numbers of $\mathcal{C}_2'$ are $h_{11}' = 3$ and $h_{12}' = 55$. In M-theory, we have moved into the moduli space of

$$\mathcal{G}_2' = \frac{\mathcal{C}_2' \times S^1}{w' \mathcal{I}} \,, \tag{4.8}$$

where the involution takes the form [20]

$$w' : \begin{cases} z_i \to \bar{z}_i \\ \xi_1 \to -\bar{\xi}_2 \,, \quad \xi_2 \to \bar{\xi}_1 \end{cases} \qquad \text{and} \qquad \mathcal{I} : \ x^{10} \to -x^{10} \,. \tag{4.9}$$

The orbifold is freely acting, hence the spectrum is given by the Betti numbers b_2' and b_3' of $\mathcal{G}_2'$. From eq. (3.20), since $h_{11}'^{-} = 0$,[33] we have

$$59 \text{ chiral multiplets} \qquad \text{and} \qquad \text{no gauge group} \,. \tag{4.10}$$

Similarly, in the orientifold description on $\mathcal{C}_2'/w'$, there are no fixed points and thus no orientifold sixplane and D6 branes. From eq. (4.5), the spectrum is also consisting of 59 chiral multiplets.

Locally, the transition from $\mathcal{G}_2$ (for $\phi > 1$) to $\mathcal{G}_2'$ amounts to a flop $S^3 \to (\mathbb{CP}^1 \times S^1)/w\mathcal{I} \cong \mathbb{RP}^3$ for each SU(2) factor. In the IIA orientifold limit $R_{10} \to 0$, it is translated into a conifold transition $S^3 \to \mathbb{RP}^2$ with no RR flux for each SO(4) factor, as was the case between the local models of eqs. (2.15) and (2.16). This describes confinement for each $\mathrm{SU}(2)^2 \cong \mathrm{SO}(4)$ factor from both points of view. However, in the compact case,

[32] Actually, only freely acting cases were considered in [48] but their arguments apply to singular orientifolds as well.

[33] Two of the (1,1) forms were already odd on $\mathcal{C}_2/w$ and the third one dual to the $\mathbb{CP}^1$ parametrized by ξ_1/ξ_2 is also odd under w'.

due to the change in the Hodge numbers of the underlying CY, the number of neutral chiral multiplets also varies in the transition. This was discussed in detail in [20, 22] and interpreted as a change of branch in the scalar potential. Actually, passing from $\mathcal{C}_2'/w'$ to $\mathcal{C}_2/w$ in type IIA orientifolds, the number of flat directions in the potential changes at the conifold point due to the appearance of additional massless states. These states are black hole chiral multiplets associated with membranes wrapped on the vanishing $\mathbb{RP}^2$'s, as in the $\mathcal{N}=2$ case [53, 54].

Let us return now to the model based on $\mathcal{G}_1$. From M-theory, brane and field theory points of view, the transitions of section 3 concerned the non-abelian gauge groups only, since both the number of neutral chiral multiplets (89 in our example) and U(1) vector multiplets (0 in our example) were constant.[34] In the description in terms of type IIA orientifold on $\mathcal{C}_1/w$, let us focus as an example on the transition II$\to$I. As discussed in section 3.2, locally this should amount to a flop $S^3 \to 0 \to S^3/w \cong \mathbb{RP}^3$, as it is the case for the non-compact model $T^*(S^3)/w$, when passing from $\mu > 0$ to $\mu < 0$. To describe quantitatively confinement in these cases, quantum corrections have to be taken into account. To compute them, it should a priori be possible to map the system in the phase with D6 branes and an O6 plane on S^3 to a type IIB description on the mirror CY with D5 branes and an O5 plane on an isolated curve. This would be a strategy in the spirit of [49, 50, 33], applied to SLAG's of various topologies and in presence of orientifold planes. Another strategy was also proposed in [51] and relates threefolds with D-branes to fourfolds. However, we reviewed in section 2.2 that two D6 branes (with their mirror partners) and an O6 plane on S^3 is dual to an $\mathbb{RP}^2$ with no RR flux in an orientifold of type IIA [27]. One can then ask if in the compact case there would be a similar closed string dual description in the sense of [10, 27], involving $\mathbb{RP}^2$ and describing the same massless neutral chiral and abelian vector multiplets?

Thus, we are looking for a type IIA orientifold on $\tilde{\mathcal{C}}_1/\tilde{w}$, where $\tilde{\mathcal{C}}_1$ is a CY admitting an antiholomorphic involution $\tilde{w}$. Let $\tilde{h}_{11}^{\pm}$ be the number of even and odd 2-forms on $\tilde{\mathcal{C}}_1$ under $\tilde{w}$, and $\tilde{h}_{12}$ the number of 3-forms. From eq. (4.5), in order to have an equal number of U(1) vector multiplets and neutral chiral multiplets on $\tilde{\mathcal{C}}_1/\tilde{w}$ and $\mathcal{C}_1/w$, we need $\tilde{h}_{11}^+ = h_{11}^+$ and $1+\tilde{h}_{12}+\tilde{h}_{11}^- = 1+h_{12}+h_{11}^-$. Taking the sum of these equations, we thus have

$$\tilde{h}_{11}^+ = h_{11}^+ \qquad \text{and} \qquad \tilde{h}_{11}+\tilde{h}_{12} = h_{11}+h_{12}\,. \tag{4.11}$$

The second of these equations is suggestive. Actually, if one supposes that there could be a general rule for relating a generic CY orientifold to a dual CY orientifold, the latter could hardly involve anything but the original manifold or its mirror. Now, since the dual type IIA orientifold should locally involve $\mathbb{RP}^2$ instead of S^3, we are led to ask whether an orientifold of type IIA on a CY threefold could be dual to an orientifold of type IIA on the mirror CY. In the mirror CY, there should be isolated $\mathbb{CP}^1$'s mirror to the 3-spheres on which branes and orientifold planes are wrapped.[35] Since $\tilde{w}$ is required to act on these

[34]Actually, one chiral multiplet parametrizes the path followed in moduli space, while the other chiral and abelian vector multiplets are relegated to the role of spectators.

[35]In general, a SLAG is mirror to an holomorphic cycle. Here, we consider the case where the holomorphic cycle is an isolated 2-sphere.

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$\mathbb{CP}^1$'s so that they become $\mathbb{RP}^2$'s, as in eqs. (2.34) and (4.9), w' is freely acting. Hence, there are no O5 planes and, due to the vanishing of the RR charge, no D5 branes. As expected to describe confinement, the non-abelian gauge group of the original orientifold does not show up in the present one.

Also, we note that for the mirror $\widetilde{\mathrm{CY}}$ of a threefold, the orientifold of type IIA on $\widetilde{\mathrm{CY}}/\tilde{w}$ is obtained in the limit $R_{10} \to 0$ of M-theory on $\tilde{\mathcal{G}} = (\widetilde{\mathrm{CY}} \times S^1)/\tilde{w}\mathcal{I}$. M-theory on the two backgrounds $\mathcal{G} = \left(\mathrm{CY} \times S^1\right)/w\mathcal{I}$ and $\tilde{\mathcal{G}}$ has an equal number of abelian vector and neutral chiral massless multiplets. In addition, locally, the S^3's of A_1 singularities in $\mathcal{G}$ are replaced by $(\mathbb{CP}^1 \times S^1)/\tilde{w}\mathcal{I} \cong \mathbb{RP}^3$ in $\tilde{\mathcal{G}}$, as was the case at the beginning of this section. Thus, $\mathcal{G}$ and $\tilde{\mathcal{G}}$ could also be dual.

Finally, one can consider more general type IIA orientifolds that involve an even number N_c of D6 branes on top of an $\mathrm{O6}_{\pm}$ wrapped on S^3 in a CY. Then, this configuration might be related to an orientifold of type IIA on the mirror CY with $(N_c/2) \pm 2$ units of RR flux on $\mathbb{RP}^2$. Similarily, the mirror statement in type IIB would be equivalent.

4.2 Non-abelian Higgsing, scalar potential and conifold transitions

We have seen that for orientifolds of compact CY's, confinement can be described by a conifold transition. We would like to see now that in a different set up of branes and orientifolds, a conifold transition can also describe a non-abelian Higgs mechanism. In all these cases, these effects are combined with a change of branch in the scalar potential.

In [23], the Higgsing of $\mathcal{N} = 2$ U(1) vector multiplets in type II compactifications on CY manifolds in terms of confinement of magnetic flux was studied. In that work, 3-spheres that vanish at some conifold locus in complex structure moduli space were considered. In general, these 3-spheres are not independent in homology. Instead, there classes satisfy linear combinations that vanish. Due to these linear relations, a 3-cycle that meets such an S^3 must meet at least another one.

Let us consider now the case illustrated in figure 13*a*. There are two S^3's intersecting a third 3-cycle at a point. An equal even number N_c of D6 branes on top of an $\mathrm{O6}_-$ plane are wrapped on the 3-spheres. The non-abelian gauge group is then $\mathrm{SO}(N_c) \times \mathrm{SO}(N_c)$. Since the S^3's are fixed by the orientifold projection, the 3-cycle that intersect them is non-orientable. When it is $\mathbb{RP}^3$, we can go to a locus in complex structure moduli space where it vanishes, as in figure 13*b*. Then, blowing up the singularity to a 2-cycle, the 3-spheres have become 3-chains connected along their boundary, a blow up 2-sphere, as can be seen in figure 13*c*. The brane system then consists of N_c D6 branes and an $\mathrm{O6}_-$ plane wrapped on the single S^3, which is the connected sum of the two original 3-spheres. Therefore, the gauge group has become the diagonal $\mathrm{SO}(N_c)$ and the whole transition is a non-abelian Higgs mechanism. Note that simultaneously, as in the case of confinement treated previously, the number of massless neutral chiral multiplets has also changed due to the variation of Hodge numbers of the CY through the conifold transition.

To conclude, we signal that an example of the above conifold transition is realized in the model based on $\mathcal{C}_3/w$, where $\mathcal{C}_3$ is a CY threefold living in a sub-family of $\mathbb{CP}^4_{11222}$ [8].

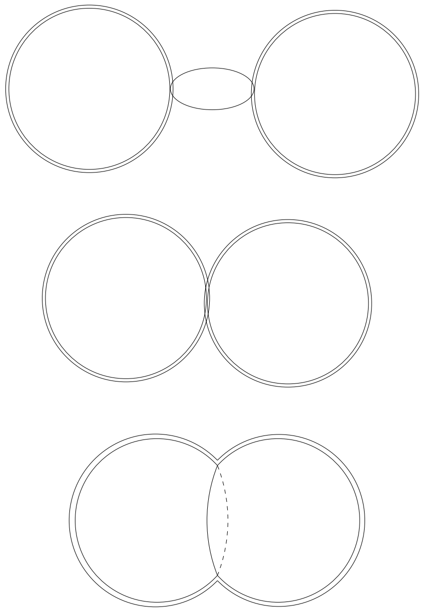

Figure 13: A conifold transition describing the Higgs mechanism $\mathrm{SO}(N_c) \times \mathrm{SO}(N_c) \to \mathrm{SO}(N_c)$.

The defining polynomial of $\mathcal{C}_3$ is

$$p_3 \equiv z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \psi_1 z_6^2 z_2^4)^2 + (z_4^2 - \psi_2 z_6^2 z_2^4)^2 + (z_5^2 - \psi_3 z_6^2 z_2^4)^2 = 0\,, \quad (4.12)$$

where ϕ and $\psi_{1,2,3}$ are real, while w acts as usually as $z_i \to \bar{z}_i$. The relevant transition occurs at $\psi_2, \psi_3 > \psi_1 = \sqrt{\phi^2 - 1}$.

5. Transitions involving 3-cycles with $b_1 > 0$

Up to now, we have considered transitions in M-theory that involve 3-cycles with vanishing first Betti number only. We would like now to consider in the same spirit 3-cycles with non-trivial b_1. In section 5.1, we shall deal with transitions where $b_1 = 1$ on both sides of the transitions. On the contrary, we shall focus in section 5.2 on an example where b_1 of the 3-cycles changes at the transition. In the first (second) case, the transitions are expected to be at infinite (finite) distance in moduli space.

5.1 Replacing S^3's by $S^2 \times S^1$'s

Let us consider models involving 3-cycles of topology $S^2 \times S^1$. In particular, we would like to see what the transitions of 3-spheres in section 3 become when each S^3 is replaced by an $S^2 \times S^1$. Following [20], one can start from the model based on $\mathcal{C}_1$ and take an orbifold $(z_4, z_5) \to (-z_4, -z_5)$ on it that is blown up. The CY manifold $\mathcal{C}_4$ obtained this way has Hodge numbers $h_{11} = 3$ and $h_{12} = 55$ and is defined by a polynomial

$$p_4 \equiv z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \psi z_2^4 z_6^2)^2 + z_7^2(z_4^4 + z_5^4) = 0\,, \quad (5.1)$$

for complex ϕ and ψ, where there are three $\mathbb{C}^*$ scaling actions, whose weights are

$$\begin{array}{c|ccccccc} & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \\ \hline \mathbb{C}_1^* & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ \mathbb{C}_2^* & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ \mathbb{C}_3^* & 0 & 0 & 0 & 1 & 1 & 0 & -2 \end{array} \qquad (5.2)$$

The forbidden set in the ambient space is now

$$(z_1, z_2) \neq (0,0)\,, \qquad (z_4, z_5) \neq (0,0) \quad \text{and} \quad (z_3, z_6) \neq (0,0)\,. \qquad (5.3)$$

There are singular points that occur in charts where z_2 and z_6 do not vanish, so that they can be rescaled to one. These singularities arise for specific values of ϕ and ψ:

$$\begin{array}{llll} \text{for any } \psi, & \text{at } \phi = +1 & : & (i^k, 1, \pm\sqrt{\psi}, z_4, z_5, 1, 0)\,, \quad (k = 0, \dots, 3)\,, \\ & \text{or } \phi = -1 & : & (i^k e^{i\pi/4}, 1, \pm\sqrt{\psi}, z_4, z_5, 1, 0)\,, \ (k = 0, \dots, 3)\,, \\ \multicolumn{2}{l}{\text{and for any } \phi, \text{ at } \psi = \pm\sqrt{\phi^2 - 1}} & : & (\phi^{1/4} i^k, 1, 0, z_4, z_5, 1, 0)\,, \quad (k = 0, \dots, 3)\,, \end{array} \qquad (5.4)$$

where the projective coordinates $(z_4, z_5) \equiv (\lambda z_4, \lambda z_5)$ $(\lambda \in \mathbb{C}_3^*)$ parametrize a $\mathbb{CP}^1$.

We now restrict ϕ, ψ to real values in order to consider the G_2 orbifold

$$\mathcal{G}_4 = \frac{\mathcal{C}_4 \times S^1}{w\mathcal{I}}\,, \qquad (5.5)$$

where the involution is

$$w: \ z_i \to \bar{z}_i \quad (i = 1, \dots, 7)\,, \qquad \mathcal{I}: \quad x^{10} \to -x^{10}\,. \qquad (5.6)$$

Looking for the fixed points of this involution, eq. (3.8) is modified to

$$x_1^4 = \phi \pm \sqrt{\phi^2 - \left[1 + (x_3^2 - \psi)^2 + x_7^2(x_4^4 + x_5^4)\right]}\,, \qquad (5.7)$$

where the real projective coordinates $(x_4, x_5) \equiv (l x_4, l x_5)$ $(l \in \mathbb{R}^*)$ parametrize a circle. Choosing a point (x_4, x_5) on this circle, eq. (5.7) can be written in terms of the variables $X_{1,3}$ defined in eq. (3.10) and $X_7 = x_7\sqrt{x_4^4 + x_5^4}$:

$$X_1^2 + X_3^2 + X_7^2 = \phi^2 - 1\,, \qquad (5.8)$$

which is an S^2 of radius $\sqrt{\phi^2 - 1}$ when $\phi \geq 1$. Since the rest of the discussion is identical to what was done after eq. (3.11), we know that each S^3 of A_1 singularities of radius $\sqrt{\phi^2 - 1}$ in $\mathcal{G}_1$ is replaced in $\mathcal{G}_4$ by a cycle $S^2 \times S^1$ of A_1 singularities where the radius of S^2 is $\sqrt{\phi^2 - 1}$. Similarly, two S^3's a distance $2\left[\psi - \sqrt{\phi^2 - 1}\right]^{1/2} > 0$ away in $\mathcal{G}_1$ become two S^2's a distance $2\left[\psi - \sqrt{\phi^2 - 1}\right]^{1/2} > 0$ away in $\mathcal{G}_4$, altogether multiplied by S^1. As a result, for $\phi < 1$, the fixed point set $\Sigma \times \{0, \pi R_{10}\}$ is empty, while for $\phi \geq 1$, Σ is composed of two identical copies Σ_+ and Σ_- of topology summarized as follows:

$$\begin{array}{lll} \psi > \sqrt{\phi^2 - 1}\,, & \Sigma_+ = (S^2 \times S^1) \cup (S^2 \times S^1) & \text{(disconnected union)}\,, \\ \psi = \sqrt{\phi^2 - 1}\,, & \Sigma_+ = (S^2 \cup S^2) \times S^1 & \text{(with a singular } S^1 \text{ of intersection)}\,, \\ |\psi| < \sqrt{\phi^2 - 1}\,, & \Sigma_+ = (S^2 \# S^2) \times S^1 \cong S^2 \times S^1 & \text{(with a connected sum of } S^2\text{'s)}\,, \\ \psi = -\sqrt{\phi^2 - 1}\,, & \Sigma_+ = \{(\phi^{1/4}, 1, 0, x_4, x_5, 1, 0)\} & \text{(a singular } S^1)\,, \\ \psi < -\sqrt{\phi^2 - 1}\,, & \Sigma_+ = \emptyset & \text{(no fixed point)}\,. \end{array} \qquad (5.9)$$

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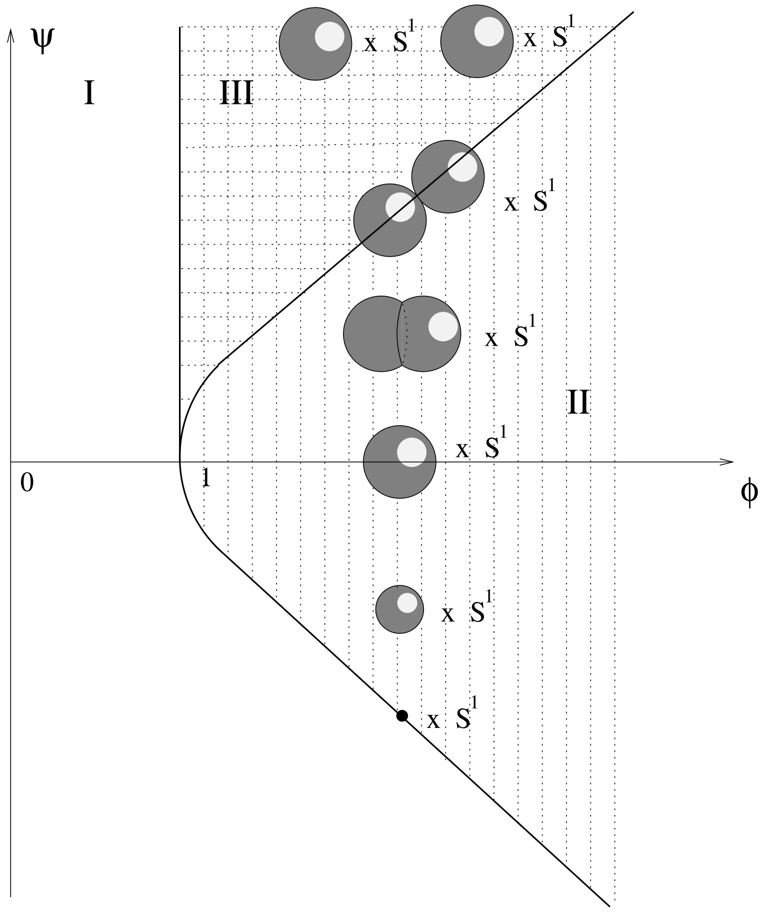

Figure 14: Phase diagram of the model based on $\mathcal{G}_4$. Σ_+ is represented for various values of (ϕ, ψ). When non-empty, it is composed of 2-spheres times circles. The transitions are not expected to be physical and should occur at infinite distance in moduli space. Hatches indicate that we sit in the Coulomb branch parametrized by axes orthogonal to the figure.

In figure 14, the fixed point set $\Sigma_+ \times \{0\}$ in $\mathcal{G}_4$ is represented for various values of the parameters. We describe now the M-theory spectrum in each phase.

– **Spectrum in phase I: $\phi < 1$ or $\phi \geq 1, \psi < -\sqrt{\phi^2 - 1}$.** In this case, since the involution is freely acting, the massless spectrum is given by eq. (3.20). Since $d(z_4/z_5) \wedge d(\bar{z}_4/\bar{z}_5)$ is odd under w, the blow up $\mathbb{CP}^1$ at $(z_4, z_5) = (0, 0)$ in $\mathcal{C}_4$ is also odd. Thus $h_{11}^+ = 0$ and we have

$$59 \text{ chiral multiplets} \qquad \text{and} \qquad \text{no gauge group}\,. \tag{5.10}$$

– **Spectrum in phase II: $\phi > 1$, $|\psi| < \sqrt{\phi^2 - 1}$.** We still have the previous spectrum associated to the invariant forms on the orbifold. In addition, for each $S^2 \times S^1$, if we do not desingularize the A_1 singularity, there is an SU(2) gauge group with $b_1(S^2 \times S^1) = 1$ chiral multiplet in the adjoint representation [11, 20]. Because $b_1 > 0$, blowing up the A_1

singularity is expected to be possible.[36] This amounts to going into the Coulomb branch of SU(2) by giving a VEV to the adjoint matter field. In this branch, SU(2) is broken to U(1) and the matter in the adjoint of SU(2) gives rise to a single massless neutral chiral multiplet. In phase II, we have at most four singular $S^2 \times S^1$'s that can be desingularized independently. Therefore, the total spectrum takes one on the following forms:

$$\begin{gathered}\text{an } \mathrm{SU}(2)^{4-k} \times \mathrm{U}(1)^k \text{ gauge group }, \quad (k=0,\ldots,4)\,, \\ \text{with 1 adjoint chiral multiplet of } \mathrm{SU}(2)^{4-k} \\ \text{and } 59+k \text{ neutral chiral multiplets }.\end{gathered} \tag{5.11}$$

In figure 14, the hatches in phase II signal that as soon as we sit in the Coulomb branch of an SU(2) gauge group, we are actually at a point in moduli space with a non-zero coordinate in a direction orthogonal to the (ϕ, ψ) plane.

– **Spectrum in phase III: $\phi > 1$, $\psi > \sqrt{\phi^2 - 1}$.** The only difference between phase III and phase II is that there are eight instead of four $S^2 \times S^1$'s of A_1 singularities that can be blown up. As a result, the spectrum is now

$$\begin{gathered}\text{an } \mathrm{SU}(2)^{8-k} \times \mathrm{U}(1)^k \text{ gauge group }, \quad (k=0,\ldots,8)\,, \\ \text{with 1 adjoint chiral multiplet of } \mathrm{SU}(2)^{8-k} \\ \text{and } 59+k \text{ neutral chiral multiplets }.\end{gathered} \tag{5.12}$$

As before, phase III is hatched in figure 14 to indicate that at fixed ϕ and ψ, we can still move in the Coulomb branches.

Brane interpretation. To understand what is happening at the transitions, we would like to translate them to the language of brane constructions in type IIA.

Let us start by reducing M-theory on $\mathcal{G}_4$ in phase II, when we sit at the origin of the Coulomb branches (i.e. $k = 0$ in eq. (5.11)). Sending R_{10} to zero, the set $\Sigma_+ \times \{0, \pi R_{10}\}$ of A_1 singularities gives rise to a single copy of $S^2 \times S^1$ with an O6 plane coincident with two D6 branes (and their mirror partners) wrapped on it.[37] In type IIA orientifold language, this system generates an $\mathrm{SO}(4) \cong \mathrm{SU}(2)^2$ gauge group with $b_1(S^2 \times S^1) = 1$ adjoint field. Since this $S^2 \times S^1$ lives in a one complex dimensional moduli space of SLAG's, the two D6 branes can slide on any other 3-cycle in this family. This describes the Coulomb branches $\mathrm{SO}(4) \to \mathrm{SU}(2) \times \mathrm{U}(1)$ or $\mathrm{SO}(4) \to \mathrm{U}(1)^2$ corresponding to $k \neq 0$ in eq. (5.11).

After two T-dualities, the local configuration is described by the type IIA system of figure 15: Two parallel NS branes orthogonal to an O4 plane together with two D4 branes sitting at $x^4 + ix^5 = v_{1,2}$ and their mirror images at $-v_{1,2}$ (see (2.1) for conventions). This describes an $\mathcal{N} = 2$ SO(4) vector multiplet in the Coulomb branch of the theory. However, the usual classical gauge coupling $\tau_{\rm cl}$ is promoted in our case to a full dynamical $\mathcal{N} = 1$

[36] under some additional circumstances [17].

[37] We again only consider the component Σ_+ of Σ; the discussion for the second one Σ_- is independent and similar.

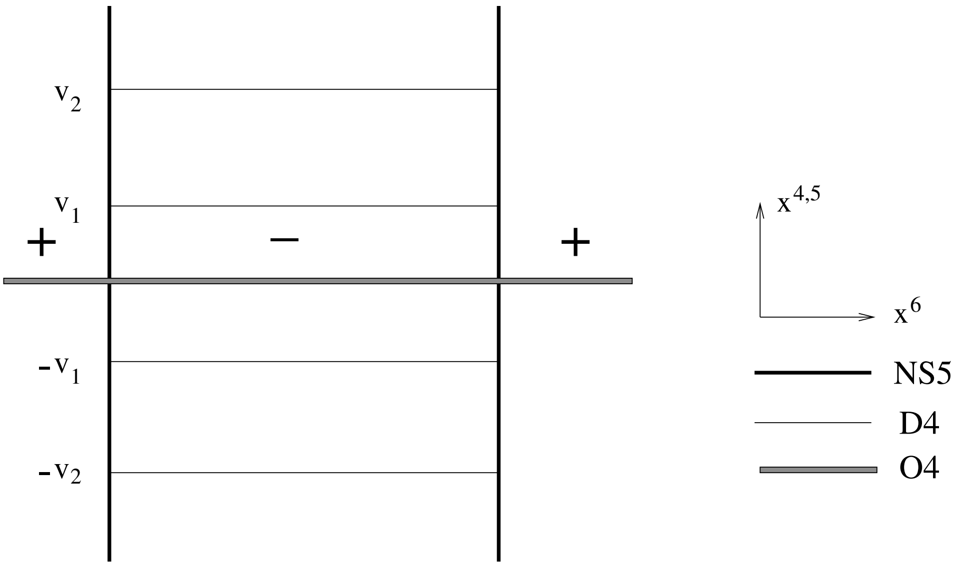

Figure 15: Brane realization of $\mathcal{N} = 2$ SO(4) SYM in the Coulomb branch.

chiral field whose VEV is a flat direction. Geometrically, its scalar component is associated with $\psi + \sqrt{\phi^2 - 1}$, the inverse gauge coupling, complexified by the M-theory 3-form, the theta angle. In phase II, $\psi + \sqrt{\phi^2 - 1} > 0$ and quantum corrections should imply that we remain in the Coulomb branches parametrized by v_1 (v_2) for the first (second) SU(2) factor [52].

If ψ decreases, when $\psi + \sqrt{\phi^2 - 1} = 0$ in figure 14, the two NS branes are coincident and we are at infinite classical gauge coupling (see figure 16d). For $\psi + \sqrt{\phi^2 - 1} < 0$ we have passed into phase I, where the gauge theory and adjoint matter have disappeared. In the brane picture, this phase is realized by separating the two NS branes in the directions $x^{7,8,9}$, while keeping them parallel so that they are mirror to each other with respect to the O4 plane, as indicated on figure 16e. Thus, phases II and I are described by parallel NS branes separated in different directions. It amounts to sending a classical parameter to infinity, the gauge coupling, and then passing into a different theory. Also, as we reviewed in section 2.2, systems of parallel NS branes are mapped in type IIB geometry to ALE spaces such as eq. (2.6), and can be desingularized in only one way given in eq. (2.7). We thus expect the transition II$\to$I to occur at infinite distance in moduli space, not corresponding to a physical transition such as a Higgs mechanism. In other words, when we are in phase II, quantum effects forbid us to leave the Coulomb branches $v_{1,2} \neq 0$ and we are an infinite distance away from the boundary domain wall $\psi = -\sqrt{\phi^2 - 1}$ where the moduli space dimension jumps. Notice that in [20], such an infinite distance away transition was associated to an $S^2 \times S^1$ flop transition, where only S^2 vanishes.

Let us consider now the transition III$\to$II. In phase III of figure 14, the brane picture is consisting of two sets of branes similar to figure 15 along a common O4 plane. The two systems of parallel Neveu-Schwarz fivebranes are non-mutually parallel, as shown in figure 16a. When ψ approaches $\sqrt{\phi^2 - 1}$, the distance L_1 between these systems decreases.

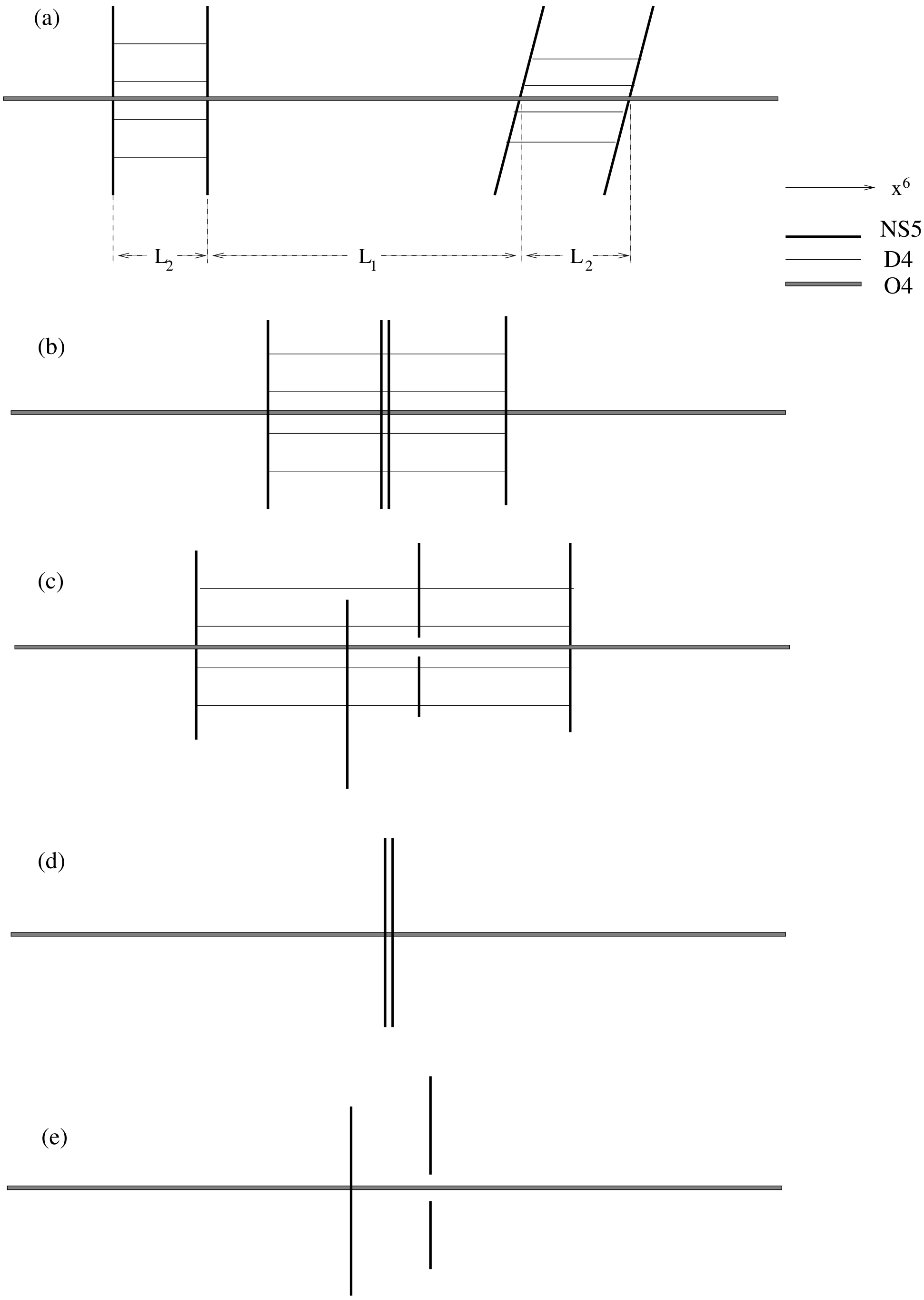

Figure 16: (a) Two SO(4) systems of branes with adjoint fields approach each other. (b) The middle branes coincide. (c) Then, they separate. (d) The left over NS branes approach each other till they coincide. (e) Finally, they separate.

Also, we expect their relative angle to decrease, so that at the transition $\psi = \sqrt{\phi^2 - 1}$ the two central NS branes are parallel and collide. This has to be the case due to the fact that the geometrical transitions $\psi \overset{>}{\to} \sqrt{\phi^2 - 1}$ and $\psi \overset{<}{\to} -\sqrt{\phi^2 - 1}$ are related into each other by the map $(\psi, z_6) \to (-\psi, iz_6)$ in the defining polynomial eq. (5.1). From the geometric point of view carracterized by the Neveu-Schwarz branes only, they are therefore identical. At the origin of the Coulomb branch (i.e. $k = 0$ in eq. (5.12)), the low energy theory is an $\mathcal{N} = 1$ $\mathrm{SO}(4)\times\mathrm{SO}(4)$ SYM with a massless chiral multiplet in the adjoint and a massive $Q \in (4, 4)$ chiral multiplet.[38]. When $L_1 = 0$, the two central NS branes are coincident (see figure 16b) and we pass into phase II by separating them in the directions $x^{7,8,9}$ (see figure 16c). As before, a brane transition that consists of approaching, colliding and separating two parallel NS branes is expected to be unphysical. Hence, the line $\psi = \sqrt{\phi^2 - 1}$ should be at infinite distance in moduli space.

Finally, the transition III→I in figure 14 is similar to the transition II→I. To summarize, the model based on $\mathcal{G}_4$ should give rise to three distinct components of moduli space. Therefore, we have separated the different domains in figure 14 by continuous lines.

5.2 3-cycles transitions with non-constant b_1

In this section, our aim is to describe other types of transitions where the first Betti number of the involved 3-cycles change. As will be precised at the end of this section, they should occur at finite distance in moduli space.

We start again with a two-parameter sub-family of manifolds $\mathcal{C}_5$ within $\mathbb{CP}^4_{11222}[8]$, whose defining polynomial is

$$p_5 \equiv z_6^4(z_1^8 + z_2^8 - 2\phi z_1^4 z_2^4) + (z_3^2 - \psi z_6^2 z_2^4)^2 + (z_4^2 - \psi z_6^2 z_2^4)^2 + z_5^4 = 0\,, \tag{5.13}$$

where ϕ and ψ are complex. This family has members where singular points occur:[39]

$$\begin{array}{llcll} \text{for any } \psi, & \text{at } \phi = +1 & : & (i^k, 1, \pm\sqrt{\psi}, \pm\sqrt{\psi}, 0, 1)\,, & (k = 0, \ldots, 3)\,, \\ & \text{or } \phi = -1 & : & (i^k e^{i\pi/4}, 1, \pm\sqrt{\psi}, \pm\sqrt{\psi}, 0, 1)\,, & (k = 0, \ldots, 3)\,, \\ \text{and for any } \phi, & \text{at } \psi = \pm\sqrt{\phi^2 - 1} & : & (i^k\phi^{1/4}, 1, i^l(\phi^2-1)^{1/4}, 0, 0, 1)\,, & (k, l = 0, \ldots, 3), \\ & & & (i^k\phi^{1/4}, 1, 0, i^l(\phi^2-1)^{1/4}, 0, 1)\,, & (k, l = 0, \ldots, 3)\,, \\ & \text{or } \psi = \pm\sqrt{\frac{\phi^2-1}{2}} & : & (i^k\phi^{1/4}, 1, 0, 0, 0, 1)\,, & (k = 0, \ldots, 3)\,, \end{array} \tag{5.14}$$

where the $+/-$ signs are independent.

We proceed by restricting ϕ and ψ to real values in order to consider the G_2 orbifold

$$\mathcal{G}_5 = \frac{\mathcal{C}_5 \times S^1}{w\mathcal{I}}\,, \tag{5.15}$$

where the involution $w\mathcal{I}$ is defined as in eq. (3.6). The orbifold point set $\Sigma \times \{0, \pi R\}$ is characterized by the special lagrangian 3-cycle Σ of w-invariant points in $\mathcal{C}_5$ given by:

$$x_6^4(x_1^8 + x_2^8 - 2\phi x_1^4 x_2^4) + (x_3^2 - \psi x_6^2 x_2^4)^2 + (x_4^2 - \psi x_6^2 x_2^4)^2 + x_5^4 = 0\,, \tag{5.16}$$

where the unknowns are real and x_2, x_6 can be scaled to 1, thanks to eq. (3.3). Solving for x_1^4, one obtains

$$x_1^4 = \phi \pm \sqrt{\phi^2 - [1 + (x_3^2 - \psi)^2 + (x_4^2 - \psi)^2 + x_5^4]}\,, \tag{5.17}$$

[38]The transition can also be considered when we sit in the Coulomb branch, as actually is described in figure 16.

[39]The determinant of second derivatives at these singularities has one vanishing eigenvalue.

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which implies $\phi \geq 1$ for having solutions. In the variables

$$X_1 = x_1^4 - \phi\,, \quad X_j = x_j^2 - \psi \ \ (j = 3, 4)\,, \quad X_5 = x_5^2 \ \mathrm{sign}(x_5)\,, \tag{5.18}$$

we find again a 3-sphere

$$X_1^2 + X_3^2 + X_4^2 + X_5^2 = \phi^2 - 1\,. \tag{5.19}$$

However, as for $\mathcal{C}_1$, X_5 as a function of x_5 is one-to-one, while X_1 as a function of x_1 is two-to-one so that Σ has again two disconnected components Σ_+, Σ_-.

Finally, we have to find the solutions for the variables $x_{3,4}$. From the inequalities $-\sqrt{\phi^2 - 1} \leq x_{3,4}^2 - \psi \leq \sqrt{\phi^2 - 1}$, one has:

- For $\psi > \sqrt{\phi^2 - 1}$: There exist two disjoint sets of solutions for x_j $(j = 3, 4)$

$$\begin{aligned} &0 < (\psi - \sqrt{\phi^2 - 1})^{1/2} \leq x_j \leq (\psi + \sqrt{\phi^2 - 1})^{1/2} \\ \text{or} \quad &- (\psi + \sqrt{\phi^2 - 1})^{1/2} \leq x_j \leq -(\psi - \sqrt{\phi^2 - 1})^{1/2} < 0\,, \end{aligned} \tag{5.20}$$

so that

$$\text{for } \psi > \sqrt{\phi^2 - 1} \quad : \quad \Sigma_+ = \bigcup_{i=1}^{4} S^3\,, \tag{5.21}$$

where the S^3's are disconnected (see figure 17*a*).

- For $\psi = \sqrt{\phi^2 - 1}$: The four previous S^3's intersect at four points $(\phi^{1/4}, 1, \pm(\phi^2 - 1)^{1/4}, 0, 0, 1)$ and $(\phi^{1/4}, 1, 0, \pm(\phi^2 - 1)^{1/4}, 0, 1)$ so that Σ_+ is connected with four singular points (see figure 17*b*). Note that the CY is also singular at these points (see eq. (5.14)).

- For $\sqrt{\frac{\phi^2-1}{2}} < \psi < \sqrt{\phi^2 - 1}$: The cyclic connected sum of the four 3-spheres has now become smooth. An important remark is that the first Betti number of Σ_+ has undergone a transition since we have now $b_1(\Sigma_+) = 1$. Topologically Σ_+ has become $S^2 \times S^1$ (see figure 17*c*):

$$\text{for } \sqrt{\tfrac{\phi^2-1}{2}} < \psi < \sqrt{\phi^2 - 1} \quad : \quad \Sigma_+ = \mathop{\#}_{i=1}^{4} S^3 \cong S^2 \times S^1\,. \tag{5.22}$$

- For $\psi = \sqrt{(\phi^2 - 1)/2}$: The size of the S^1 vanishes and Σ_+ can be recognized as a 3-sphere with its north and south poles identified. The coordinates of this singular point are $(\phi^{1/4}, 1, 0, 0, 0, 1)$, where the CY is also singular (see figure 17*d*).

- For $-\sqrt{(\phi^2 - 1)/2} < \psi < \sqrt{(\phi^2 - 1)/2}$: The previously identified north and south poles are now distinct. As a result, we have now

$$\text{for } -\sqrt{\tfrac{\phi^2-1}{2}} < \psi < \sqrt{\tfrac{\phi^2-1}{2}} \quad : \quad \Sigma_+ = S^3\,, \tag{5.23}$$

whose first Betti number is back to $b_1(\Sigma_+) = 0$ (see figures 17*e*,*f*,*g*).

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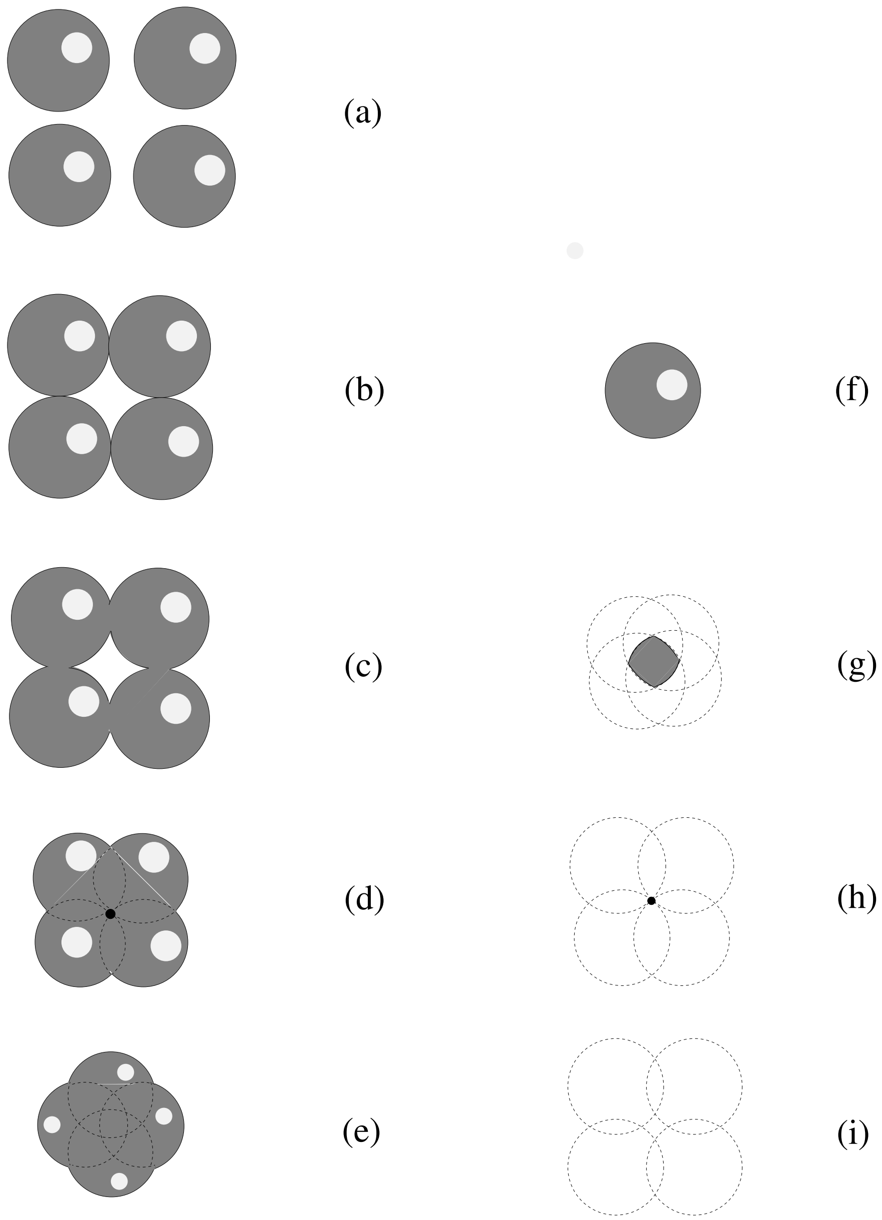

Figure 17: (a) Σ_+ is composed of four 3-spheres that apprach each other. (b) They intersect at four singular points of the manifold. (c) They are connected so that topologically they are equivalent to a single $S^2 \times S^1$. (d) the S^1 has shrunk to a singular point of the manifold. (e,f,g) Σ_+ has become an S^3. (h) This S^3 has shrunk to a singular point of the manifold. (i) Σ_+ is empty.

- For $\psi = -\sqrt{(\phi^2-1)/2}$: The size of the S^3 vanishes and Σ_+ has collapsed to a point, so that

$$\text{for } \psi = -\sqrt{\tfrac{\phi^2-1}{2}} \quad : \quad \Sigma_+ = \{(\phi^{1/4}, 1, 0, 0, 0, 1)\}\,, \tag{5.24}$$

where the CY is singular (see figure 17h).

- Finally for $\psi < -\sqrt{(\phi^2-1)/2}$: There is no solution for $x_{3,4}$ and we have (see figure 17*i*)

$$\text{for } \psi < -\sqrt{\tfrac{\phi^2-1}{2}} \quad : \qquad \Sigma_+ = \emptyset\,. \tag{5.25}$$

To summarize, the topology of the fixed point set Σ_+ takes the form, according to ψ and $\phi > 1$,

$$\begin{array}{lll}
\psi > \sqrt{\phi^2-1}\,, & \Sigma_+ = \bigcup_{i=1}^4 S^3 & \text{(disconnected sum)}\,, \\
\psi = \sqrt{\phi^2-1}\,, & \Sigma_+ = \bigcup_{i=1}^4 S^3 & \text{(intersecting at four singular} \\
\sqrt{\frac{\phi^2-1}{2}} < \psi < \sqrt{\phi^2-1}\,, & \Sigma_+ = \#_{i=1}^4 S^3 \cong S^2\times S^1 & \qquad\qquad\qquad\text{points)}\,, \\
\psi = \sqrt{\frac{\phi^2-1}{2}}\,, & \Sigma_+ = S^3 & \text{(north and south poles} \\
|\psi| < \sqrt{\frac{\phi^2-1}{2}}\,, & \Sigma_+ = S^3 & \qquad\qquad\qquad\text{identified)}\,, \\
\psi = -\sqrt{\frac{\phi^2-1}{2}}\,, & \Sigma_+ = \{(\phi^{1/4},1,0,0,0,1)\} & \text{(one singular point)}\,, \\
\psi < -\sqrt{frac\phi^2-12}\,, & \Sigma_+ = \emptyset\,. &
\end{array} \tag{5.26}$$

Finally, we have represented in figure 18 the associated phase diagram. Let us determine now the massless spectrum.

– **Spectrum in phase I:** $\phi < 1$ **or** $\phi \geq 1$, $\psi < -\sqrt{\frac{\phi^2-1}{2}}$**.** In this phase the orbifold is smooth and the spectrum is determined by eq. (3.20) where $h_{11}^+ = 0$. Thus, there are

$$89 \text{ chiral multiplets} \qquad \text{and} \qquad \text{no gauge group}\,. \tag{5.27}$$

– **Spectrum in phase II:** $\phi > 1$, $|\psi| < \sqrt{\frac{\phi^2-1}{2}}$**.** This phase is similar to phase II of the model based on $\mathcal{G}_1$. In M-theory, there two copies of $\Sigma_+ \cup \Sigma_- = S^3 \cup S^3$ of A_1 singularities, one at $x^{10} = 0$ and the other at $x^{10} = \pi R_{10}$. Hence, we have

$$1 \text{ vector multiplet of } \mathrm{SU}(2)^4 \qquad \text{and} \qquad 89 \text{ neutral chiral multiplets}\,. \tag{5.28}$$

– **Spectrum in phase III:** $\phi > 1$, $\sqrt{\frac{\phi^2-1}{2}} < \psi < \sqrt{\phi^2-1}$**.** As in phase II of the model based on $\mathcal{G}_4$, Σ_+ is composed of a single connected component that gives rise to an SU(2) gauge group with one adjoint chiral field. In total, since we have four copies of $S^2 \times S^1$ in $\mathcal{G}_5$ and as we can go independently in the Coulomb branch of each SU(2) factor, the massless spectrum consists of

$$\begin{gathered}
\text{an } \mathrm{SU}(2)^{4-k} \times \mathrm{U}(1)^k \text{ gauge group}\,, \quad (k = 0,\ldots,4)\,, \\
\text{with 1 adjoint chiral multiplet of } \mathrm{SU}(2)^{4-k} \\
\text{and } 59+k \text{ neutral chiral multiplets}\,.
\end{gathered} \tag{5.29}$$

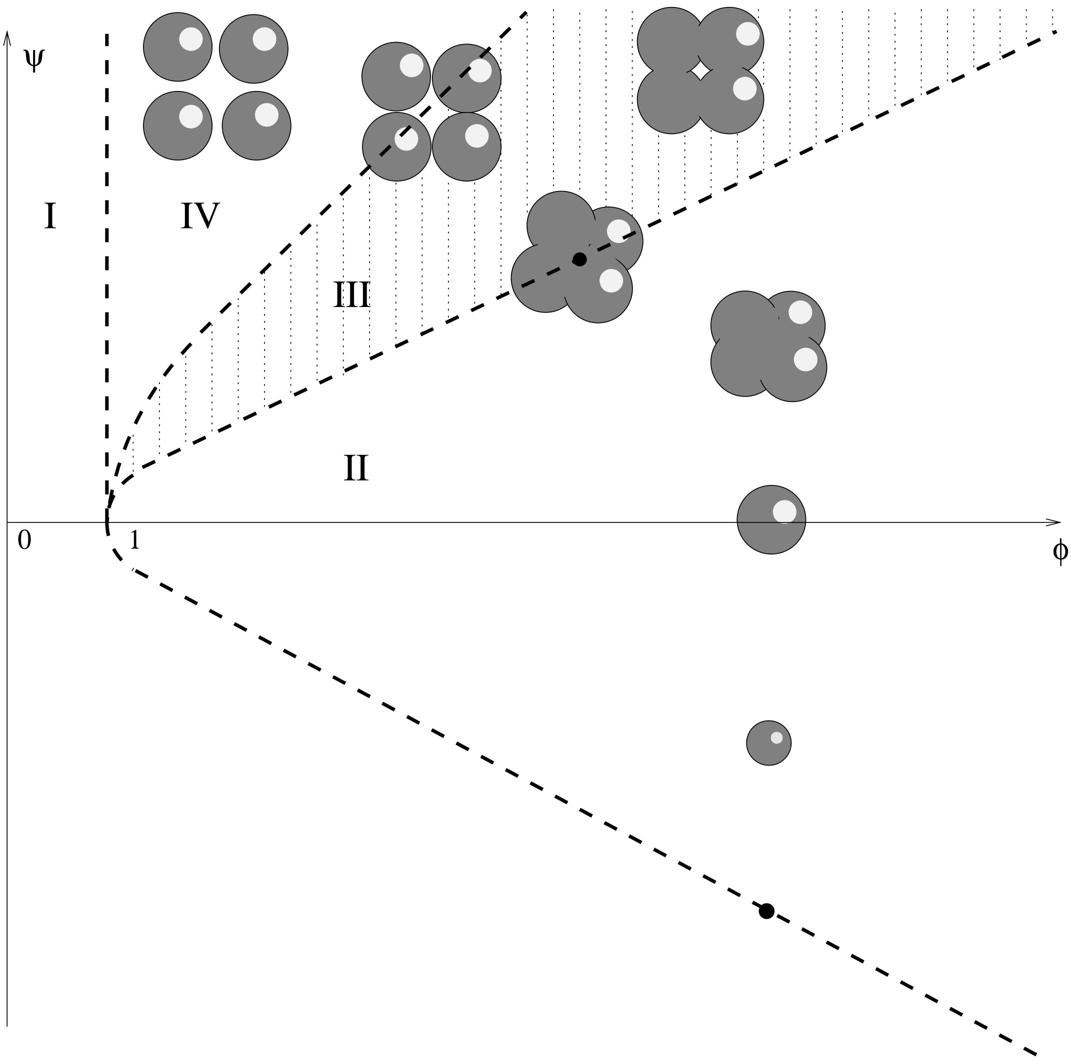

Figure 18: Phase diagram of the model based on $\mathcal{G}_5$. Σ_+ is represented for various values of (ϕ, ψ). It is composed of 3-spheres in phases II and IV. In Phase III, four 3-spheres are connected so that topologically $\Sigma_+ = S^2 \times S^1$. The transitions should occur at finite distance in moduli space. Hatches indicate the presence of a Coulomb branch parametrized by axes orthogonal to the figure.

– **Spectrum in phase IV:** $\phi > 1$, $\psi > \sqrt{\phi^2 - 1}$. In this phase, Σ_+ is composed of four disconnected S^3's so that we have

$$1 \text{ vector multiplet of } \mathrm{SU}(2)^{16} \qquad \text{and} \qquad 89 \text{ neutral chiral multiplets}\,. \tag{5.30}$$

Partial interpretation. As already said in footnote 39, at each singularity of eq. (5.14), the determinant of second derivatives vanishes. In fact, if one replaces z_5^4 in the defining polynomial p_5 in eq. (5.13) by $(z_5^2 - \psi_3 z_6^2 z_2^4)^2$ as in eq. (4.12), then the transitions become standard conifold transitions. Explicitely, switching on ψ_3 has the effect of separating the vanishing 3-spheres that are coincident when $\psi_3 = 0$. Thus, for $\psi_3 \neq 0$ the transitions are at finite distance in moduli space [55], a fact that should still be valid in the case $\psi_3 \equiv 0$ we studied in order to avoid irrelevant complications in the discussion.[40].

Actually, there is no new effect concerning the transitions II→I and IV→I associated to confinement. As in figure 9, they are represented with dashed lines in figure 18. To understand what is happening at the transition IV→III, it would be usefull to map the M-theory geometry to a brane system of NS and D4 branes. Unfortunately, we have not been able to determine the dual brane picture in details. However, for Σ_+ in phase IV, it could involve four copies of the system of figure 10, with a common O4 plane along a compact x^6 axis. The gauge group is then $SO(4)^4$ as in M-theory. When ψ approaches $\sqrt{\phi^2 - 1}$, the NS brane of each system approaches the NS' brane of the adjacent system till they intersect. At this stage, there are four copies of intersecting orthogonal Neveu-Schwarz fivebranes along the compact x^6 direction. Then, each pair of intersecting branes bend, as in figure 4, and we are left with two D4 branes and their mirrors along an O4 plane on x^6. This bending causes the Higgsing of the gauge group to a single SO(4) due to non-perturbative effects. However, in this brane picture, the system is now locally $\mathcal{N} = 4$, i.e. with three massless chiral multiplets in the adjoint of SO(4), while the M-theory geometry predicts only a local $\mathcal{N} = 2$ system, i.e. with one adjoint. Thus, there must be some additional ingredients in the brane picture breaking $\mathcal{N} = 4 \to \mathcal{N} = 2$. Also, these ingredients must be of first importance for describing the transition III→II, where only the massless adjoint matter has disappeared. Clearly, such transitions where the first Betti number of 3-cycles varies need to be further understood.

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Note Added: As this article was being completed, we received the paper [56], which describes in detail Seiberg duality from type IIB geometric set up.

[40] As a particular case, if one considers the transitions for $\psi_3 \equiv \psi$, one finds that there are in particular eight S^3's centered at the corners of a cube that are approaching each other before they are connected. As in our case, the first Betti number of Σ_+ jumps to a non trivial value $b_1 > 0$.

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